ECE 301 Solution to Homework Assignment 2

1. Indicate whether the following systems are causal, invertible, linear, memoryless, and/or time invariant by circling the correct options. (A system may have more than one of these properties.) Justify your answer.

(a) \( y(t) = x(t-2) + x(2-t) \) (causal, invertible, \underline{linear}, memoryless, \underline{time invariant})

Causality: The system is NOT causal because for any \( t < 1 \), the output depends on a future input, e.g. \( y(0) = x(-2) + x(2) \).

Invertibility: The system is NOT invertible because, if \( w(t) \) is even, then the output \( y(t) \) corresponding to \( x_1(t) = w(t)u(t) \) is indistinguishable from that corresponding to \( x_2(t) = w(t)u(-t) \).

Linearity: The system is linear because if

\[
\begin{align*}
y_1(t) &= x_1(t-2) + x_1(2-t), \quad (1) \\
y_2(t) &= x_2(t-2) + x_2(2-t), \quad \text{and} \quad (2) \\
x(t) &= \alpha x_1(t) + \beta x_2(t), \quad (3)
\end{align*}
\]

then the output \( y(t) \) corresponding to the input \( x(t) \) is

\[
\begin{align*}
y(t) &= x(t-2) + x(2-t) \quad (4) \\
     &= (\alpha x_1 + \beta x_2)(t-2) + (\alpha x_1 + \beta x_2)(2-t) \quad (5) \\
     &= \alpha x_1(t-2) + \beta x_2(t-2) + \alpha x_1(2-t) + \beta x_2(2-t) \quad (6) \\
     &= \alpha (x_1(t-2) + x_1(2-t)) + \beta (x_2(t-2) + x_2(2-t)) \quad (7) \\
     &= \alpha y_1(t) + \beta y_2(t). \quad (8)
\end{align*}
\]

Memorylessness: The system is NOT memoryless because the output at time \( t \) depends on input values at times other than \( t \).

Time Invariance: The system is time invariant. To see this, we let \( y(t) \) be the output corresponding to the input \( x(t) \) and let \( x_a(t) = x(t-a)^1 \). Then the output \( y_a(t) \) corresponding to the input signal \( x_a(t) \) is

\[
\begin{align*}
y_a(t) &= x_a(t-2) + x_a(2-t) \quad (9) \\
     &= x((t-a)-2) + x(2-(t-a)) \quad (10) \\
     &= y(t-a). \quad (11)
\end{align*}
\]

(b) \( y(t) = \int_{-\infty}^{2} x(\tau)d\tau \) (causal, invertible, \underline{linear}, memoryless, time invariant)

---

1The character “\( a \)” on the left hand side of the equals sign is a label. The variable \( a \) on the right hand side represents an arbitrary real number. It may be easier to think of a specific number, say, 3, so long as you note that for the system to be time invariant, the test must work for any real value, not just 3.
Causality: The system is NOT causal because for any $t < 2$, the output depends on a future input.

Invertibility: The system is NOT invertible because only the integral and not the input function itself determines the output, e.g. the output corresponding to input $x_1(t) = u(t) - u(t - 1)$ is the same as that corresponding to input $x_2(t) = 2[u(t) - u(t - 1/2)]$.

Linearity: The system is linear because if

\[ y_1(t) = \int_{-\infty}^{2} x_1(\tau)d\tau, \text{ and } y_2(t) = \int_{-\infty}^{2} x_2(\tau)d\tau, \tag{12} \]

and $x(t) = \alpha x_1(t) + \beta x_2(t)$, then the output $y(t)$ corresponding to the input $x(t)$ is

\[ y(t) = \int_{-\infty}^{2} (\alpha x_1 + \beta x_2)(\tau)d\tau \tag{14} \]

\[ = \int_{-\infty}^{2} \alpha x_1(\tau)d\tau + \int_{-\infty}^{2} \beta x_2(\tau)d\tau \tag{15} \]

\[ = \alpha y_1(t) + \beta y_2(t). \tag{16} \]

Memorylessness: The system is NOT memoryless because the output at time $t$ depends on input values at times other than $t$.

Time Invariance: The system is NOT time invariant. To see this, we let

\[ x(t) = u(t) - u(t - 1) \] so that \[ y(t) = \int_{-\infty}^{2} x(\tau)d\tau \tag{17} \]

\[ = \int_{0}^{1} d\tau \tag{18} \]

\[ = 1, \tag{19} \]

i.e. constant for all $t$, so in particular, $y(t - 3) = 1$ for any value of $t$\textsuperscript{2}. However, if we let

\[ x_3(t) = x(t - 3) = u(t - 3) - u(t - 4) \]

. Then

\[ y_3(t) = \int_{-\infty}^{2} x_3(\tau)d\tau \tag{21} \]

\[ = \int_{3}^{2} x(\tau)d\tau \tag{22} \]

\[ = 0 \tag{23} \]

\[ \neq y(t - 3). \tag{24} \]

\textsuperscript{2}We need only a single delay value for which the property does not hold to show that the system is not time invariant, so 3 will do.
(c) \( y(t) = (dx/dt)(t) \)  

**Causality:** The system is memoryless, hence causal.

**Invertibility:** The system is NOT invertible because any two inputs that differ by a constant yield the same output.

**Linearity:** The system is linear because if

\[
\begin{align*}
y_1(t) &= (dx_1/dt)(t), \\
y_2(t) &= (dx_2/dt)(t),
\end{align*}
\]

and \( x(t) = \alpha x_1(t) + \beta x_2(t) \), then the output \( y(t) \) corresponding to the input \( x(t) \) is

\[
\begin{align*}
y(t) &= (d(\alpha x_1 + \beta x_2)/dt)(t) \\
&= \alpha(dx_1/dt)(t) + \beta(dx_2/dt)(t) \\
&= \alpha y_1(t) + \beta y_2(t).
\end{align*}
\]

**Memorylessness:** The system is NOT memoryless because the output at time \( t \) depends on input values at times other than \( t \).

**Time Invariance:** The system is time invariant. To see this, we let \( y(t) \) be the output corresponding to the input \( x(t) \) and let \( x_a(t) = x(t-a) \). Then the output \( y_a(t) \) corresponding to the input signal \( x_a(t) \) is

\[
y_a(t) = (dx_a/dt)(t) = (d(x)/dt)(t-a) = y(t-a).
\]

(d) \( y(t) = x(t/3) \)  

**Causality:** The system is NOT causal because if \( t < 0 \), then the output depends on future values of the input, e.g. for \( t = 3 \) we have \( y(-3) = x(-1) \).

**Invertibility:** The system is invertible by applying the function \( w(t) = y(3t) \).

**Linearity:** The system is linear because if

\[
\begin{align*}
y_1(t) &= x_1(t/3), \\
y_2(t) &= x_2(t/3)
\end{align*}
\]

and \( x(t) = \alpha x_1(t) + \beta x_2(t) \), then the output \( y(t) \) corresponding to the input \( x(t) \) is

\[
\begin{align*}
y(t) &= (\alpha x_1 + \beta x_2)(t/3) \\
&= \alpha x_1(t/3) + \beta x_2(t/3) \\
&= \alpha y_1(t) + \beta y_2(t)
\end{align*}
\]

**Memorylessness:** The system is NOT memoryless because the output at time \( t \) depends on input values at times other than \( t \).
**Time Invariance:** The system is time invariant. To see this, we let $y(t)$ be the output corresponding to the input $x(t)$ and let $x_a(t) = x(t-a)$. Then the output $y_a(t)$ corresponding to the input signal $x_a(t)$ is

$$y_a(t) = x_a(t/3) = x((t-a)/3) = y(t-a)$$

(e) $y(t) = \cos(x(t))$  
\[\text{causal, invertible, linear, memoryless, time invariant}\]

**Causality:** The system is memoryless, hence causal.

**Invertibility:** The system is NOT invertible, e.g. suppose that $x_1(t) = (\pi/2)u(t)$ and $x_2(t) = -(\pi/2)u(t)$. Then $y_1(t) = \cos(x_1(t)) = 0 = \cos(x_2(t)) = y_2(t), \forall t$.

**Linearity:** The system is NOT linear because if

$$y_1(t) = \cos(x_1(t)), \text{ and}$$

$$y_2(t) = \cos(x_2(t)),$$

and $x(t) = \alpha x_1(t) + \beta x_2(t)$, then the output $y(t)$ corresponding to the input $x(t)$ is

$$y(t) = \cos(\alpha x_1(t) + \beta x_2(t))$$

$$= \cos(\alpha x_1(t)) \cos(\beta x_2(t)) - \sin(\alpha x_1(t)) \sin(\beta x_2(t))$$

$$\neq \alpha \cos(x_1(t)) + \beta \cos(x_2(t))$$

in general. (We used an identity from Chapter B of the textbook to get the second equality above.)

**Memorylessness:** The system is memoryless because the output at time $t$ depends on input values at only time $t$.

**Time Invariance:** The system is time invariant. To see this, we let $y(t)$ be the output corresponding to the input $x(t)$ and let $x_a(t) = x(t-a)$. Then the output $y_a(t)$ corresponding to the input signal $x_a(t)$ is

$$y_a(t) = \cos(x_a(t)) = \cos(x(t-a)) = y(t-a).$$
2. For the system described by the differential equation

\[(D^2 + \alpha D + 6)y(t) = (D + 4)x(t)\]  \hspace{1cm} (43)

(a) Suppose \(\alpha = 5\).

i. Find the eigenvalues and modes.

**Solution:** Solving \(Q(\lambda) = \lambda^2 + 5\lambda + 6 = (\lambda + 2)(\lambda + 3) = 0\), we obtain the eigenvalues \(\lambda_1 = -2\) and \(\lambda_2 = -3\). The corresponding modes are \(c_1 e^{-2t}\) and \(c_2 e^{-3t}\).

ii. Find the zero input response \(y_0(t)\) corresponding to the initial conditions \(y(0) = 1, \dot{y}(0) = 3\).

**Solution:** The ZIR is

\[y_0(t) = c_1 e^{-2t} + c_2 e^{-3t}, \hspace{0.5cm} \text{so} \hspace{1cm} (44)\]
\[\dot{y}_0(t) = -2c_1 e^{-2t} - 3c_2 e^{-3t}. \hspace{0.5cm} (45)\]

Solving

\[y_0(0) = 1 = c_1 + c_2 \hspace{0.5cm} (46)\]
\[\dot{y}_0(0) = 3 = -2c_1 - 3c_2 \hspace{0.5cm} (47)\]

for \(c_1\) and \(c_2\) yields

\[c_1 = 6 \hspace{0.5cm} \text{and} \hspace{0.5cm} c_2 = -5\]

so

\[y_0(t) = 6e^{-2t} - 5e^{-3t}\]

iii. Find the impulse response \(h(t)\).

**Solution:** First we solve for \(y_n(t)\) by applying the initial conditions \(y(0) = 0\) and \(\dot{y}(0) = 1\) to the linear combination of the modes

\[y_n(t) = c_1 e^{-2t} + c_2 e^{-3t}.\]

Solving

\[y_n(0) = 0 = c_1 + c_2 \hspace{0.5cm} (48)\]
\[\dot{y}_n(0) = 1 = -2c_1 - 3c_2 \hspace{0.5cm} (49)\]

for \(c_1\) and \(c_2\) yields

\[c_1 = 1 \hspace{0.5cm} \text{and} \hspace{0.5cm} c_2 = -1\]

so

\[y_n(t) = e^{-2t} - e^{-3t}, \hspace{0.5cm} \text{and} \hspace{1cm} (50)\]
\[\dot{y}_n(t) = -2e^{-2t} + 3e^{-3t}. \hspace{0.5cm} (51)\]
The impulse response will be given by
\[ h(t) = b_0 \delta(t) + [P(D)y_n(t)]u(t). \] (52)

The degree of \(Q(D)\) being one greater than the degree of \(P(D)\), the value of \(b_0\) is zero. We obtain
\[ h(t) = [(D + 4)y_n(t)]u(t) \] (53)
\[ = [-2e^{-2t} + 3e^{-3t} + 4(e^{-2t} - e^{-3t})]u(t) \] (54)
\[ = [2e^{-2t} - e^{-3t}]u(t). \] (55)

(b) Now suppose \(\alpha = 1\).

i. Find the eigenvalues and modes.

Solution: Solving \(Q(\lambda) = \lambda^2 + \lambda + 6 = (\lambda + \alpha + j\beta)(\lambda + \alpha - j\beta) = 0\), where \(\alpha = -1/2\) and \(\beta = \sqrt{23}/2\) we obtain the eigenvalues \(\lambda_1 = -1/2 + j\sqrt{23}/2\)
and \(\lambda_2 = -1/2 - j\sqrt{23}/2\). The corresponding modes are \(c_1 e^{\lambda_1 t}\) and \(c_2 e^{\lambda_2 t}\), or \(ce^{\alpha t}\cos(\beta t + \theta)\), depending upon which representation one prefers.

ii. Find the zero input response \(y_0(t)\) corresponding to the initial conditions \(y(0) = 1, \dot{y}(0) = 3\).

Solution: For readability, I’ll write the intermediate equations in terms of \(\alpha\) and \(\beta\) and substitute for them at the end. The ZIR is
\[ y_0(t) = c_1 e^{(\alpha + j\beta)t} + c_2 e^{(\alpha - j\beta)t}, \quad \text{so} \]
\[ \dot{y}_0(t) = (\alpha + j\beta)c_1 e^{(\alpha + j\beta)t} + (\alpha - j\beta)c_2 e^{(\alpha - j\beta)t}. \] (57)

Solving
\[ y_0(0) = 1 = c_1 + c_2 \quad \text{and} \]
\[ \dot{y}_0(0) = 3 = (\alpha + j\beta)c_1 + (\alpha - j\beta)c_2 \] (59)
for \(c_1\) and \(c_2\) yields
\[ c_1 = \frac{\beta - j(3 - \alpha)}{2\beta} \quad \text{and} \quad c_2 = \frac{\beta + j(3 - \alpha)}{2\beta} \] (60)

so
\[ y_0(t) = \frac{\beta - j(3 - \alpha)}{2\beta} e^{(\alpha + j\beta)t} + \frac{\beta + j(3 - \alpha)}{2\beta} e^{(\alpha - j\beta)t} \] (61)
\[ = e^{\alpha t} \left( \frac{e^{j\beta t} + e^{-j\beta t}}{\beta} \right) + \frac{j(3 - \alpha)e^{\alpha t}}{\beta} \left( \frac{-e^{j\beta t} + e^{-j\beta t}}{2} \right) \] (62)
\[ = e^{\alpha t} \cos(\beta t) + \frac{(3 - \alpha)e^{\alpha t}}{\beta} \sin(\beta t) \] (63)
\[ = e^{-t/2} \cos(\sqrt{23}t/2) + \frac{7e^{-t/2}}{\sqrt{23}} \sin(\sqrt{23}t) \] (64)
iii. Find the impulse response $h(t)$.

**Solution:** First we solve for $y_n(t)$ by applying the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$ to the linear combination of the modes

$$y_n(t) = c_1 e^{(\alpha+j\beta)t} + c_2 e^{(\alpha-j\beta)t}$$

Solving

$$y_n(0) = 0 = c_1 + c_2 \quad \text{and} \quad \dot{y}_n(0) = 1 = (\alpha + j\beta)c_1 + (\alpha - j\beta)c_2$$

for $c_1$ and $c_2$ yields

$$c_1 = -j\frac{1}{2\beta} \quad \text{and} \quad c_2 = j\frac{1}{2\beta}$$

so

$$y_n(t) = j\frac{1}{2\beta}(-e^{(\alpha+j\beta)t} + e^{(\alpha-j\beta)t}) \quad \text{(67)}$$

$$= e^{\alpha t} \left(\frac{\sin(\beta t)}{\beta}\right), \quad \text{and} \quad \text{(68)}$$

$$\dot{y}_n(t) = \alpha e^{\alpha t} \left(\frac{\sin(\beta t)}{\beta}\right) + e^{\alpha t} \cos(\beta t) \quad \text{(69)}$$

The impulse response is then

$$h(t) = b_0 \delta(t) + [P(D)y_n(t)]u(t). \quad \text{(70)}$$

The degree of $Q(D)$ being one greater than the degree of $P(D)$, the value of $b_0$ is zero. We obtain

$$h(t) = [(D + 4)y_n(t)]u(t) \quad \text{(71)}$$

$$= \left[(\alpha + 4)e^{\alpha t} \left(\frac{\sin(\beta t)}{\beta}\right) + e^{\alpha t} \cos(\beta t)\right]u(t) \quad \text{(72)}$$

$$= 7e^{-t/2} \left(\frac{\sin(\sqrt{23}t)}{\sqrt{23}}\right) + e^{-t/2} \cos(\sqrt{23}t). \quad \text{(73)}$$

(c) Finally, suppose $\alpha = 2\sqrt{6}$.

i. Find the eigenvalues and modes.

**Solution:** Solving $Q(\lambda) = \lambda^2 + 2\sqrt{6}\lambda + 6 = (\lambda + \sqrt{6})^2 = 0$, we obtain the eigenvalues $\lambda_1 = \lambda_2 = -\sqrt{6}$. The corresponding modes are $c_1 e^{-\sqrt{6}t}$ and $c_2 te^{-\sqrt{6}t}$. 
ii. Find the zero input response $y_0(t)$ corresponding to the initial conditions $y(0) = 1$, $\dot{y}(0) = 3$.

**Solution:** The ZIR is

$$y_0(t) = c_1 e^{-\sqrt{6} t} + c_2 t e^{-\sqrt{6} t}, \quad \text{so}$$

$$\dot{y}_0(t) = -\sqrt{6}(c_1 e^{-\sqrt{6} t} + c_2 t e^{-\sqrt{6} t}) + c_2 e^{-\sqrt{6} t}. \quad (74) \quad (75)$$

Solving

$$y_0(0) = 1 = c_1$$

$$\dot{y}_0(0) = 3 = -\sqrt{6}c_1 + c_2 \quad (76) \quad (77)$$

for $c_1$ and $c_2$ yields

$$c_1 = 1 \quad \text{and} \quad c_2 = 3 + \sqrt{6}$$

so

$$y_0(t) = (1 + (3 + \sqrt{6})t)e^{-\sqrt{6} t}. \quad (78) \quad (79)$$

iii. Find the impulse response $h(t)$.

**Solution:** First we solve for $y_n(t)$ by applying the initial conditions $y(0) = 0$ and $\dot{y}(0) = 1$ to the linear combination of the modes

$$y_n(t) = (c_1 + c_2 t)e^{-\sqrt{6} t}. \quad (80)$$

Solving

$$y_n(0) = 0 = c_1$$

$$\dot{y}_n(0) = 1 = -\sqrt{6}c_1 + c_2 \quad (81) \quad (82)$$

for $c_1$ and $c_2$ yields

$$c_1 = 0 \quad \text{and} \quad c_2 = 1$$

so

$$y_n(t) = te^{-\sqrt{6} t}, \quad \text{and}$$

$$\dot{y}_n(t) = -\sqrt{6}t e^{-\sqrt{6} t} + e^{-\sqrt{6} t} \quad (83) \quad (84)$$

The impulse response is then

$$h(t) = b_0 \delta(t) + [P(D)y_n(t)]u(t). \quad (85)$$

The degree of $Q(D)$ being one greater than the degree of $P(D)$, the value of $b_0$ is zero. We obtain

$$h(t) = [(D + 4)y_n(t)]u(t) \quad (86) \quad (87)$$

$$= [(-\sqrt{6}t + 1)e^{-\sqrt{6} t} + 4te^{-\sqrt{6} t}]u(t) \quad (88) \quad (89)$$

$$= [(1 + (-\sqrt{6} + 4)t)e^{-\sqrt{6} t}]u(t). \quad (90) \quad (91)$$