1. For the system described by the difference equation

$$y[n] + 5y[n - 1] + 6y[n - 2] = x[n - 1] + 4x[n - 2]$$  \hspace{1cm} (1)

(a) Find the unit impulse response $h[n]$.

**Solution:** Following the procedure outlined on pp. 283–284 of the textbook, we will find the unit impulse response in the form of (3.49), namely

$$h[n] = \frac{b_N}{a_N}\delta[n] + y_c[n]u[n].$$  \hspace{1cm} (2)

To determine we first write the difference equation in the form of (3.42a). To do this we apply a change of variables. Let $\tilde{n} = n - 2$. Then we have the equation

$$y[\tilde{n} + 2] + 5y[\tilde{n} + 1] + 6y[\tilde{n}] = x[\tilde{n} + 1] + 4x[\tilde{n}],$$  \hspace{1cm} (3)

which becomes

$$(E^2 + 5E + 6)y[\tilde{n}] = (E + 4)x[\tilde{n}],$$  \hspace{1cm} (4)

in the form of (3.42a). From this we see that $N = 2$ and $b_N/a_N = 4/6 = 2/3$. To obtain $y_c[n]$ we proceed as follows:

$$Q(E)y[\tilde{n}] = (E^2 + 5E + 6)y[\tilde{n}] = (E + 4)x[\tilde{n}] = P(E)x[\tilde{n}].$$  \hspace{1cm} (5)

The modes are $c_1(-2)^n$ and $c_2(-3)^n$ because factoring $Q(\gamma)$ yields characteristic equation $Q(\gamma) = \gamma^2 + 5\gamma + 6 = (\gamma + 2)(\gamma + 3) = 0$, which yields eigenvalues $\gamma_1 = -2$ and $\gamma_2 = -3$. Thus

$$y_c[n] = c_1(-2)^n + c_2(-3)^n.$$  \hspace{1cm} (6)

We find the coefficients $c_i$ by applying the initial conditions

$$h[-1] = h[-2] = 0$$  \hspace{1cm} (7)

as indicated in (3.44), page 283. Specifically, we have

$$h[n] + 5h[n - 1] + 6h[n - 2] = \delta[n - 1] + 4\delta[n - 2]$$  \hspace{1cm} (8)

which for $n = 0$ yields

$$h[0] + 5h[-1] + 6h[-2] = \delta[-1] + 4\delta[-2]$$  \hspace{1cm} (9)

or

$$h[0] = -5h[-1] - 6h[-2] + \delta[-1] + 4\delta[-2] = -0 - 5(0) + 6(0) + 4(0) = 0$$  \hspace{1cm} (10)
and for \( n = 1 \) yields
\[
h[1] + 5h[0] + 6h[-1] = \delta[0] + 4\delta[-1]
\] (11)
or
\[
h[1] = -5h[0] - 6h[-1] + \delta[0] + 4\delta[-1] = -5(0) - 6(0) + 1 + 4(0) = 1.
\] (12)

Now we must have that
\[
h[n] = (2/3)\delta[n] + c_1(-2)^n + c_2(-3)^n
\] (13)
so substituting for \( n = 0 \) and \( n = 1 \) yields a simultaneous pair of linear equations
\[
\begin{align*}
0 &= 2/3 + c_1 + c_2 \quad (= (2/3)\delta[0] + c_1(-2)^0 + c_2(-3)^0) \quad (14) \\
1 &= 0 - 2c_1 - 3c_2 \quad (= (2/3)\delta[1] + c_1(-2)^1 + c_2(-3)^1) \quad (15) \\
\end{align*}
\] (16)
which have solution \( c_1 = -1, \ c_2 = 1/3 \). Thus
\[
h[n] = (2/3)\delta[n] - (-2)^n + (1/3)c_2(-3)^n = (2/3)\delta[n] - (-2)^n - (-3)^{n-1}
\] (17)
(b) Find the zero state response corresponding to the input

\[ x[n] = u[n] - u[n - 3]. \tag{18} \]

Indicate which properties of the convolution sum you use at each step and simplify your result as much as possible.

**Solution:** We will solve this by convolution, using that

\[ x[n] = u[n] - u[n - 3] = \delta[n] + \delta[n - 1] + \delta[n - 2]. \tag{19} \]

Thus by additivity of convolution,

\[ y[n] = x[n] * h[n] = \delta[n] * h[n] + \delta[n - 1] * h[n] + \delta[n - 2] * h[n]. \tag{20} \]

Of course, the identity property ("Convolution with an Impulse" in the text) together with the shifting property yield

\[ y[n] = x[n] * h[n] = h[n] + h[n - 1] + h[n - 2]. \tag{21} \]

Thus

\[
\begin{align*}
y[n] &= \left( \frac{2}{3} \right) \delta[n] - (-2)^n - (-3)^{n-1} \\
&+ \left( \frac{2}{3} \right) \delta[n - 1] - (-2)^{n-1} - (-3)^{n-2} \\
&+ \left( \frac{2}{3} \right) \delta[n - 2] - (-2)^2 - (-3)^{n-3}
\end{align*}
\] \tag{22}

Simplifying, we obtain

\[
\begin{align*}
y[n] &= \left( \frac{2}{3} \right) (u[n] - u[n - 3]) \delta[n] - \left( (-2)^n + (-2)^{n-1} + (-2)^{n-2} \right) \\
&- \left( (-3)^{n-1} + (-3)^{n-2} + (-3)^{n-3} \right)
\end{align*}
\] \tag{23}

and finally

\[
\begin{align*}
y[n] &= \left( \frac{2}{3} \right) (u[n] - u[n - 3]) - (4 - 2 + 1) (-2)^{n-2} - (9 - 3 + 1) (-3)^{n-3} \\
&= \left( \frac{2}{3} \right) (u[n] - u[n - 3]) - 3(-2)^{n-2} - 7(-3)^{n-3}
\end{align*}
\] \tag{24}

2. For the system described by the differential equation

\[ (D^2 + 5D + 6)y(t) = (D + 4)x(t) \tag{25} \]

(a) Find the impulse response \( h(t) \).

**Solution:** The answer to this part of the problem was given in the solution to Problem 1a of the previous homework, namely

\[ h(t) = [2e^{-2t} - e^{-3t}]u(t). \tag{26} \]
(b) Find the transfer function $H(s)$.

The transfer function $H(s)$ can be obtained either by noting that

$$H(s) = \frac{P(s)}{Q(s)} = \frac{s + 4}{s^2 + 5s + 6} \tag{27}$$

or by noting that

$$H(s) = \mathcal{L}\{h(t)\} = \frac{2}{s + 2} + \frac{-1}{s + 3} = \frac{2(s + 3) - (s + 2)}{s^2 + 5s + 6} = \frac{s + 4}{s^2 + 5s + 6}. \tag{28}$$

(c) Find the zero state response corresponding to the input

$$x(t) = e^{-3t}[u(t) - u(t - 2)]. \tag{29}$$

Simplify your result as much as possible.

**Solution:** We will obtain $Y(s) = X(s)H(s)$ and take the inverse transform to obtain $y(t)$. First, we rewrite $x(t)$ as

$$x(t) = e^{-3t}u(t) - e^{-6}e^{-3(t-2)}u(t - 2). \tag{30}$$

Now we have Laplace transform

$$X(s) = \frac{1}{s + 3} - \frac{e^{-2(s+3)}}{s + 3} \tag{31}$$

so

$$Y(s) = \left(\frac{s + 4}{s^2 + 5s + 6}\right)\left(\frac{1 - e^{-2(s+3)}}{s + 3}\right). \tag{32}$$

The partial fraction expansion is then

$$Y(s) = \frac{A}{s + 2} + \frac{B}{s + 3} + \frac{C}{(s + 3)^2} \tag{33}$$

so we need

$$A(s^2 + 6s + 9) + B(s^2 + 5s + 6) + C(s + 2) = (s + 4)(1 - e^{-2(s+3)}) \tag{34}$$

which simplifies to

$$A + B = 0 \tag{35}$$

$$6A + 5B + C = (1 - e^{-2(s+3)}) \tag{36}$$

$$9A + 6B + 2C = 4(1 - e^{-2(s+3)}). \tag{37}$$

Solving for $A$, $B$, and $C$ yields

$$A = 2(1 - e^{-2(s+3)}) \tag{38}$$

$$B = -2(1 - e^{-2(s+3)}) \tag{39}$$

$$C = -1(1 - e^{-2(s+3)}) \tag{40}$$
so we can write

\[ Y(s) = \left( \frac{2}{s + 2} \right) + \frac{-2}{s + 3} + \frac{-1}{(s + 3)^2} \right) - \left( \frac{2}{s + 2} + \frac{-2}{s + 3} + \frac{-1}{(s + 3)^2} \right)e^{-2(s+3)} \]

and thus

\[ y(t) = \left( 2e^{-2t} + (-2 - t)e^{-3t} \right)u(t) - \\
e^{-6} \left( 2e^{-2(t-2)} + (-2 - (t - 2))e^{-3(t-2)} \right)u(t - 2). \]

We simplify this to

\[ y(t) = \left( 2e^{-2t} - (t + 2)e^{-3t} \right)u(t) - \\
\left( 2e^{-2(t+1)} - te^{-3t} \right)u(t - 2) \]

or

\[ y(t) = 2e^{-2t}[u(t) - e^{-2}u(t - 2)] - 2e^{-3t}u(t) - \\
te^{-3t}[u(t) - u(t - 2)] \]