1. Experiments, Outcomes, and Sample Spaces:

Several experiments are described below. Note that in specifying an experiment, we first specify an operation or procedure to be carried out and then specify the method of observation (i.e., outcomes or sample space such as \( S_1 = \{ x \mid x \in \mathbb{R} \text{ and } x > 0 \} \) and \( S_2 = \{ a, e, i, o, u \} \)).

For the following experiments, identify the sample spaces (be very specific, use proper definitions of sets). (8 points):

(a) \( E_1 \): Toss a true coin and observe the “up” face.

\[ S_1 = \{ H, T \} \]

(b) \( E_2 \): Toss a true coin three times and observe the sequence of heads and tails.

\[ S_2 = \{ HHH, HHT, HTH, HTT, THH, THT, TTH, TTT \} \]

(c) \( E_3 \): Toss a coin three times and observe the total number of heads.

\[ S_3 = \{ 0, 1, 2, 3 \} \]

(d) \( E_4 \): Toss a pair of dice and observe the sum of the “up” faces.

\[ S_4 = \{ 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12 \} \]
2. Union and Intersection of Events, Probabilities, and Independence:

An electronic system consists of four components: A, B, C, and D. The operation of the system can be represented by four switches, with A and B in series, and C and D in parallel with the series combination of A and B, as shown in Figure 1 below. Continuity between input and output means the system is in operation. Assume that C and D are 95% pairwise reliable, but A and B are 99% pairwise reliable, with all failures occurring independently. (12 points).

(a) Represent the event \( \mathcal{O} = \{\text{Entire system is operating}\} \), in terms of the events
\[
\begin{align*}
A &= \{\text{A is operating}\} \\
B &= \{\text{B is operating}\} \\
C &= \{\text{C is operating}\} \\
D &= \{\text{D is operating}\}.
\end{align*}
\]

(b) Find the overall reliability of the system (i.e., the probability that the system is operating).

\[
\mathcal{O} = (A \cap B) \cup C \cup D
\]  

(a) From the diagram,

(b) Note that our textbook has not defined the term ‘pairwise reliable’. Apparently, the textbook or lecture notes used in the semester this exam was given defined the pairwise reliability as the probability that at least one of the pair will operate.

Using this definition, and given that failures occur independently, we
determine the overall reliability to be

\[ P[O] = P[(A \cap B) \cup C \cup D] \]
\[ = P[A \cap B] + P[C \cup D] - P[A \cap B]P[C \cup D] \]
\[ = 0.99 + 0.95 - 0.99(0.95) \]
\[ = 0.9995 \]
3. Discrete Random Variables, Independence, and Conditional Probabilities:

Consider a 4-bit register that stores a Binary Coded Decimal (BCD) number, with all numbers from 0 to 9 equally likely. The BCD code is shown in Table 1 below. The power supply current is $i = 0.1 + 0.05k \ \mu A$, where $k$ is the number of 1’s in the register. (15 points).

Table 1. BCD Code for Problem 3.

<table>
<thead>
<tr>
<th>No.</th>
<th>$B_4$</th>
<th>$B_3$</th>
<th>$B_2$</th>
<th>$B_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

(a) Which pair of bits, if any, are independent?

(b) Find $P[B_4 = 1|B_1 = 1]$.

(c) Find the probability that the power supply is in the range $0.20 \mu A \leq i \leq 0.30 \mu A$.

(a) We have four bits so there are $\binom{4}{2} = 6$ ways to choose a pair of bits. Reading the values of the probabilities from the table, we calculate as follows:

\[
\begin{align*}
P[B_1B_2] &= 2/10 = (5/10)(4/10) = P[B_1]P[B_2] \\
\end{align*}
\]

which indicates that the first three pairs, $B_1, B_2, B_3, B_4$, and $B_1, B_4$ are independent. (b) Let $B_i$ be the event that bit $i$ of the stored code is one. The probability of each of these events is easily determined by counting the ones in the appropriate column of the table. Then by the definition of conditional probability (D1.6),

\[
P[B_4|B_1] = \frac{P[B_4]P[B_1]}{P[B_1]} = \frac{1/10}{1/2} = 1/5
\]
(c) Examining the table we see that the number of ones in the register takes values 0, 1, 2, and 3. Given that
\[ i(k) = 0.1 + 0.05k, \]
we see that the current is 0.2 if \( k = 2 \), and 0.25 if \( k = 3 \). The codes for which the current is in the range of interest are thus those for 3, 5, 6, 9, and 7. Since the codes are equally likely, we have
\[ P[0.2 \leq i(k) \leq 0.3] = \frac{5}{10} = 1/2. \]
4. Total Probability and Bayes’ Theorem:

Three printers do work for the publications office of IUPUI. The publications office does not negotiate a contract penalty for late work, and the data below reflect a large amount of experience with these printers.

<table>
<thead>
<tr>
<th>Printer, i</th>
<th>Fraction of Contracts Held by Printer i</th>
<th>Fraction of Time Delivery More than One Month Late</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2</td>
<td>0.1</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0.4</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.2</td>
</tr>
</tbody>
</table>

A department observes that its recruiting booklet is more than a month late, what is the probability that the contract is held by printer 3? (15 points).

Define $B_i$ to be the event that the contract is held by printer $i$. From the table we see that $P[B_3] = 1/2$ and $P[\text{late}|B_3] = 1/5$.

Bayes’ Theorem, Theorem 1.11 in the text, tells us that

$$P[B_3|\text{late}] = \frac{P[\text{late}|B_3]P[B_3]}{P[\text{late}]},$$

so if we determine $P[\text{late}]$, we can then find the requested probability.

The Law of Total Probability, Theorem 1.10 in the text, allows us to calculate $P[\text{late}]$, using the appropriate values from the table. We then use the previous equation to find $P[B_3|\text{late}]$.

$$P[\text{late}] = \sum_{i=1}^{3} P[\text{late}|B_i]P[B_i] = 0.2(0.1) + 0.3(0.4) + 0.5(0.2) = 0.24$$

$$P[B_3|\text{late}] = \frac{0.2(0.5)}{0.24} = \frac{10}{24} = \frac{5}{12}$$
5. Bernoulli Trials:

A certain class of highway bridge has 12 supporting columns, each of which is inspected annually for weakness. Experience shows that, with annual inspection, the probability of a column weakening seriously during the year is $10^{-4}$, independent of the condition of the other columns. What is the probability that two bad columns would occur (10 points).

Define the random variable $B$ to be the number of bad columns. Then $B$ has a binomial distribution (see Definition 2.7) with parameters $(n, p) = (12, 10^{-4})$. Accordingly

$$P[B = 2] = P_B(2) = \binom{12}{2}p^2(1-p)^{12-2} = \frac{12(11)}{2}(0.0001)^2(0.9999)^{10}.$$ (18)
6. Discrete Random Variables:

An integer is chosen at random between 0 and 4 inclusive (i.e., \( i \) is selected from the set \( i = \{0, 1, 2, 3, 4\} \) at random). A random variable is assigned to each outcome according to the following rule

\[
X = \sin \left( \frac{i \pi}{4} \right),
\]

where \( i \) is the outcome. Find and sketch the probability mass function (PMF) for \( X \), \( P_X(x) \).

(15 points)

The random variable \( X \) is a function of the chosen integer \( i \). We assume that each integer is equally likely to be chosen, since we are not told anything more than that they are chosen ‘at random’. We then determine the PMF in the steps below.

Step 1: Determine the values that the random variable \( X \) takes for each value of \( i \).

\[
X(i) = \begin{cases} 
0 & \text{if } i = 0 \\
1/\sqrt{2} & \text{if } i = 1 \\
1 & \text{if } i = 2 \\
1/\sqrt{2} & \text{if } i = 3 \\
0 & \text{if } i = 4 \\
0 & \text{otherwise}
\end{cases}
\]

(19)

Step 2: The set of values that occur on the right hand side of the above equation is \( \{0, 1/\sqrt{2}, 1\} \). Because the \( i \) values are mutually exclusive, we can determine the probabilities of the random variable \( X \) taking a particular value \( x \) by simply adding the contributions of the probabilities of the \( i \) values that yield the value \( x \).

\[
P_X(x) = \begin{cases} 
2/5 & x = 0 \\
2/5 & x = 1/\sqrt{2} \\
1/5 & x = 1
\end{cases}
\]

(20)

Note that the question also requests a plot of the PMF. This type of plot is difficult to typeset, so since it should be obvious how to plot the PMF, I have not done so.
7. Continuous Random Variables:

Suppose the probability density function (PDF) of \( X, f_X(x) \), is

\[
 f_X(x) = \begin{cases} 
 0, & 0 \leq x < 2 \\
 \frac{3}{2} \left( \frac{x}{2} - 2 \right), & 2 \leq x < 4 \\
 3a - \frac{9}{2} x, & 4 \leq x \leq 5 \\
 0, & x > 5 
\end{cases}
\]

Determine the following: \( \text{(15 points)} \).

(a). The value of \( a \) that makes \( f_X(x) \) a valid PDF.

(b). The cumulative density function (CDF), \( F_X(x) \).

(a) Here we just integrate and solve for \( a \).

\[
\int_{-\infty}^{\infty} f_X(x) \, dx = \int_2^4 \frac{a}{2} \left( x - 2 \right) \, dx + \int_4^5 \frac{a}{2} \left( 6 - x \right) \, dx
\]

\[
= \frac{a}{2} \left( \frac{x^2}{2} - 2x \right) \bigg|_2^4 + \frac{a}{2} \left( 6x - \frac{x^2}{2} \right) \bigg|_4^5
\]

\[
= \frac{a}{2} \left[ \left( \frac{16}{2} - 8 \right) - \left( \frac{4}{2} - 4 \right) \right] + \frac{a}{2} \left[ \left( 30 - \frac{25}{2} \right) - \left( 24 - \frac{16}{2} \right) \right]
\]

\[
= \left( \frac{7}{4} \right) a = 1
\]

So \( a = 4/7 \).

(b) For simplicity, we let \( b = a/2 \) so

\[
f_X(x) = \begin{cases} 
 0, & -\infty < x < 2 \\
 b(x - 2), & 2 \leq x < 4 \\
 b(6 - x), & 4 \leq x \leq 5 \\
 0, & x > 5 
\end{cases}
\]

Let’s start with the first interval, \((-\infty, 2)\). The integral of zero is zero so we have \( F_X(x) = 0, \quad x \in (-\infty, 2) \), and we let \( F_0(x) = 0 \).

Next we integrate over the interval \([2, 4)\). We obtain

\[
F_X(x) = \int_2^x b(y - 2) \, dy = b \left( \frac{y^2}{2} - 2y \right) \bigg|_2^x + F_0(2) \quad x \in [2, 4).
\]

We simplify and call the expression we obtain \( F_1(x) \) so

\[
F_1(x) = b \left( \frac{x^2}{2} - 2x + 2 \right).
\]

(27)
Next we integrate over the interval \([4, 5]\). We obtain

\[
F_X(x) = \int_4^x b(6 - y)dy = b\left(6y - y^2/2\right)|^x_4 + F_1(4) \quad x \in [4, 5].
\] (28)

We simplify and call the expression we obtain \(F_2(x)\) so

\[
F_2(x) = b\left(-x^2/2 + 6x - 14\right).
\] (29)

Now since the PDF is zero for all \(x > 5\), we should have \(F_2(5) = 1\). We check, substituting for \(b = a/2 = 2/7\):

\[
F_2(5) = 2/7\left(-25/2 + 30 - 14\right) = 1/7\left(-25 + 60 - 28\right) = 7/7 = 1.
\] (30)

Having discovered no error, we assemble our CDF from the functions \(F_i(x)\) that we obtained on the intervals, and, of course, substitute for \(b\) to obtain our final answer,

\[
F_X(x) = \begin{cases} 
0 & \infty < x < 2 \\
\frac{2}{5}(x^2/2 - 2x + 2) & 2 \leq x < 4 \\
\frac{2}{7}(-x^2/2 + 6x - 14) & 4 \leq x \leq 5 \\
1 & 5 < x
\end{cases}
\] (31)
8. Continuous Random Variables:

Find the value of the question mark within the box. You may use drawings for ease of understanding. (10 points).

(a) Gaussian PDF
\[ \int_{b}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(x-\mu)^2} \, dx = 0.5 \]

(b) Exponential PDF
\[ \int_{b}^{\infty} \lambda e^{-\lambda x} \, dx = 0.5 \]

(a) By symmetry of the Gaussian distribution, the lower limit of the integral should be \( \mu \).

(b) To determine the lower limit of integration for this example, we have to perform the integration and solve for the correct value.

\[ \int_{b}^{\infty} \lambda e^{-\lambda x} \, dx = -e^{-\lambda x} \bigg|_{b}^{\infty} = e^{-\lambda b} = 0.5 \quad (32) \]

so
\[ -\lambda b = \ln \frac{1}{2} \quad (33) \]

and thus
\[ b = -\frac{\ln \frac{1}{2}}{\lambda} = -\frac{\ln 2^{-1}}{\lambda} = \frac{\ln 2}{\lambda} \quad (34) \]