ECE 302 Homework Assignment 3 Solution

Note: To obtain credit for an answer, you must provide adequate justification. Also, if it is possible to obtain a numeric answer, you should not only find an expression for the answer, but also solve for the numeric answer.

1. Consider a binary code with 5 bits (each of which is either 0, 1) in each code word.

   (a) How many different code words are there?
   Solution: Think of the five bits as corresponding to the outcomes of five subexperiments. Then by repeated application of the Fundamental Principle of Counting (Definition 1.11, page 28 of the text), there are \(2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^5 = 32\) possible codewords.

   (b) How many code words have exactly two zeros?
   Solution: If we have exactly two zeros, we can view the options available as the choice of locations of the two zeros within the five bits, so there are \(\binom{5}{2} = 10\) code words containing exactly two zeros.

   (c) How many code words begin with zero?
   Solution: If the first bit is zero, we have \(2^4 = 16\) ways to choose the other bits.

   (d) How many code words end in two ones?
   Solution: There are \(2^3 = 8\) ways to choose the first three bits.

2. Consider a ternary code with 5 bits (each of which is either 0, 1, or 2) in each code word.

   (a) How many different code words are there?
   Solution: With 3 options rather than 2 for each bit, the number of distinct codewords is \(3^5 = 243\).

   (b) How many code words have exactly two zeros?
   Solution: This time we must account for both the placement of the zeros and the two possible choices for the other bits so there are \(\binom{5}{2} \cdot 2^3 = 80\) code words.

   (c) How many code words begin with zero?
   Solution: There are \(3^4 = 81\) choices for the remaining bits so 81 code words with first bit zero.

   (d) How many code words end in two ones?
   Solution: There are \(3^3 = 27\) choices for the remaining bits so 27 code words with last two bits one.
3. Consider a language whose alphabet has 5 letters, “p”, “q”, “r”, “s”, and “t”.

(a) How many four-letter words can be formed using this alphabet?

**Solution:** There are 5 choices for each letter of the word so $5^4 = 625$ choices.

(b) How many four-letter words can be formed using this alphabet if no repetition is allowed?

**Solution:** If no repetition is allowed, there are 5 choices for the first letter, four for the second, three for the third, and two for the fourth, so there are $5 \cdot 4 \cdot 3 \cdot 2 = 120$ possible words.

(c) How many four-letter words can be formed from the letters “p” and “q”? 

**Solution:** With two choices for each of four letters, there are $2^4 = 16$ words.

(d) How many five-letter words can be formed using this alphabet.

**Solution:** Five letters, each of which can be one of the five elements of the alphabet, so by the Fundamental Principle of Counting, there are $5^5 = 3125$ words.

4. Consider a binary code with 6 bits in each code word. Suppose that the probability of a bit being zero is 0.7, independent of the values of any other bit.

(a) What is the probability of the code word 001111 occurring?

**Solution:** It is tempting to reason as follows: the code word 001111 is only one of the $2^6 = 64$ possible code words so the probability of 001111 occurring is $1/64$. The error in this reasoning is that because zeros and ones are not equiprobable, some code words are more likely than others. The probability of the code word 001111 occurring is thus $0.7^20.3^4 = 3969/10^6$.

(b) What is the probability that a code word contains exactly four ones?

**Solution:** The probability that the code word contains exactly four ones is $\binom{6}{4}0.7^40.3^2 = 15(3969)/10^6$.

(c) What is the probability that a code word contains 3 ones and 3 zeros?

**Solution:** The probability is $\binom{6}{3}0.7^30.3^3 = 20(9261)/10^6$.

(d) What is the probability that the first and second bits are equal, the third and fourth bits are equal, and the fifth and sixth bits are equal?

**Solution:** Let’s call this event $A$. The probability of the event will be the sum of the probabilities of the code words that meet these requirements, namely,


Note that

$$P[000011] = P[001100] + P[110000] = 0.7^4(0.3^2)$$
and

\[ P[00111] = P[11001] = P[11100] = 0.7^2(0.3^4). \]

Finally, \( P[00000] = 0.7^6 \) and \( P[11111] = 0.3^6 \), thus

\[ P[A] = 0.7^6 + 3(0.3^2)0.7^4 + 3(0.3^4)0.7^2 \approx 0.1591. \]

5. A student takes three courses in a particular semester. In each course, the student will earn an A with probability 0.1, a B with probability 0.5, or a C with probability 0.4, independently of the grade in any other course. An A is worth 4 points, a B 3, and a C 2 in calculating the student’s GPA.

(a) What is the sample space?

**Solution:** \( S = \{ AAA, AAB, ABA, BAA, BAB, BBA, BBB, AAC, ACA, ACC, CAA, CAC, CCA, CCC, BBC, BCB, BCC, CBB, CBC, CCB, ABC, ACB, BAC, BCA, CAB, CBA \}. \) (Check: \(|S| = 27 = 3^3\).)

(b) Draw a table indicating the GPA values corresponding to the sample outcomes.

**Solution:** Since we will need the probabilities of the various outcomes in order to find the PMF of the GPA, we’ll list these in the table as well. The probabilities are obtained using the matlab script shown in Figure 1.

<table>
<thead>
<tr>
<th>Outcome</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>AAA</td>
<td>12/3 = 4</td>
</tr>
<tr>
<td>AAB,ABA,BAA</td>
<td>11/3</td>
</tr>
<tr>
<td>ABB,BAB,BBA,AAC,ACA,CAA</td>
<td>10/3</td>
</tr>
<tr>
<td>BBB,ABC,ACB,BAC,BCA,CAB,CBA</td>
<td>9/3 = 3</td>
</tr>
<tr>
<td>BBC,BCB,CBB,ACC,CAC,CCA</td>
<td>8/3</td>
</tr>
<tr>
<td>BCC,CBC,BCC</td>
<td>7/3</td>
</tr>
<tr>
<td>CCC</td>
<td>6/3 = 2</td>
</tr>
</tbody>
</table>

(c) What is \( S_{GPA} \)?

**Solution:** \( S_{GPA} = \{ 4, 11/3, 10/3, 3, 8/3, 7/3, 2 \}. \)

(d) What is the PMF \( P_{GPA}(s) \)?

**Solution:** From the table, the PMF of the GPA is

\[
P_{GPA}(s) = \begin{cases} 
0.064 & s = 4 \\
0.240 & s = 11/3 \\
0.315 & s = 10/3 \\
0.126 & s = 3 \\
0.060 & s = 8/3 \\
0.120 & s = 7/3 \\
0.075 & s = 2
\end{cases}
\]

(Check: 0.064 + 0.240 + 0.315 + 0.126 + 0.060 + 0.120 + 0.075 = 1.)
%%% Problem 5

count = 0;
P = zeros(10,1);
um = zeros(10,3);
for index1 = 0:3; %%% number of A's
    for index2 = 0:3; %%% number of B's
        for index3 = 0:3; %%% number of C's
            index_sum = index1 + index2 + index3; %%% "number" of grades
            if index_sum == 3; %%% number of grades must be 3
                count = count + 1;
                if max([index1,index2,index3]) == 3;
                    P(count) = (0.1^index1)*(0.5^index2)*(0.4^index3);
                elseif max([index1,index2,index3]) == 2;
                    P(count) = 3*(0.1^index1)*(0.5^index2)*(0.4^index3);
                elseif max([index1,index2,index3]) == 1;
                    P(count) = 6*(0.1^index1)*(0.5^index2)*(0.4^index3);
                end;
                GPA(count) = (4*index1 + 3*index2 + 2*index3)/3;
            num(count,:) = [index1,index2,index3];
        end;
    end;
end;
sum(P)
num
P
GPA
eGPA = GPA’*P

Figure 1: Matlab script for computing the probabilities of the GPA's.
6. Suppose that the random variable \( N \) has PMF

\[
P_N(n) = \begin{cases} 
  c/n & n \in \{1, 2, 3, 4\} \\
  0 & \text{otherwise}
\end{cases}
\]

(a) What is the value of \( c \)?

**Solution:** In order to determine the value of \( c \), we apply one of the Axioms of probability. Noting that

\[
\sum_{n \in \{1, 2, 3, 4\}} P_N(n) = c \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) = c \left( \frac{12 + 6 + 4 + 3}{12} \right) = c \left( \frac{25}{12} \right) = 1
\]

we find that \( c = 12/25 \).

(b) What is \( P[N \geq 2] \)?

**Solution:** \( P[N \geq 2] = 1 - P[N < 2] = 1 - P[N = 1] = 1 - 12/25 = 13/25 \)

(c) What is \( P[N = 2] \)?

**Solution:** \( P[N = 2] = \left( \frac{12}{25} \right) \left( \frac{1}{2} \right) = \frac{6}{25} \).

(d) What is \( P[N \text{ even}] \)?

**Solution:** \( P[N \text{ even}] = P[N = 2] + P[N = 4] \) because the events are mutually exclusive. Thus \( P[N \text{ even}] = \left( \frac{12}{25} \right) \left( \frac{3}{4} \right) = \left( \frac{9}{25} \right) \).

7. Suppose that the random variable \( N \) has PMF

\[
P_N(n) = \begin{cases} 
  cn & n \in \{1, 2, 3, 4\} \\
  0 & \text{otherwise}
\end{cases}
\]

(a) What is the value of \( c \)?

**Solution:** Again applying one of the Axioms of probability, we find \( c(1 + 2 + 3 + 4) = 1 \) so \( c = 1/10 \).

(b) What is \( P[N \geq 2] \)?

**Solution:** Now \( P[N \geq 2] = 1 - P[N = 1] = 1 - 1/10 = 9/10 \).

(c) What is \( P[N = 2] \)?

**Solution:** \( P[N = 2] = 2/10 = 1/5 \).

(d) What is \( P[1 < N < 3] \)?

**Solution:** \( P[1 < N < 3] = P[N = 2] = 1/5 \) from part (c).

8. Suppose that a package of M&M’s, is equally likely to have 8, 9, 10, or 11, or 12 yellow ones.

(a) What is the PMF of the number of yellow M&M’s in a package?

**Solution:**

\[
P_Y(y) = \begin{cases} 
  1/5 & y \in \{8, 9, 10, 11, 12\} \\
  0 & \text{otherwise}
\end{cases}
\]
(b) What is \( P[Y < 10] \)?

**Solution:**

\[ P[Y < 10] = P[Y = 8] + P[Y = 9] = \frac{2}{5}. \]

(c) What is \( P[Y > 12] \)?

**Solution:**

\[ P[Y > 12] = 0. \]

(d) What is \( P[9 < Y < 11] \)?

**Solution:**

\[ P[9 < Y < 11] = P[Y = 10] = \frac{1}{5}. \]

9. Suppose that it rains (\( r \)) with probability \( \frac{4}{7} \) on any given day, independent of the weather on any other day. Suppose further that it snows (\( * \)) with probability \( \frac{2}{7} \) on any given day, independent of the weather on any other day. Finally, suppose that the rest of the time it is sunny (\( s \)). Assume that the weather does not change during a given day.

(a) What is the probability that it rains every day of a given week?

**Solution:**

\[ P[rrrrrrr] = \left( \frac{4}{7} \right)^7 \left( \frac{3}{7} \right)^0. \]

(b) What is the probability that it rains twice, snows twice, and is sunny three days in a given week?

**Solution:**

\[ \left( \begin{array}{c} 7 \\ 2, 2, 3 \end{array} \right) \left( \frac{4}{7} \right)^2 \left( \frac{2}{7} \right)^2 \left( \frac{1}{7} \right)^3 = 210 \left( 4^2 \cdot 2^2 \cdot 1^3 \right) / 7^7 = 210(64) / 7^7 \]

(c) What is the probability that it rains twice as many days as it snows in a given week?

**Solution:**

\[ \begin{align*}
\left( \begin{array}{c} 7 \\ 4, 2, 1 \end{array} \right) &\left( \frac{4}{7} \right)^4 \left( \frac{2}{7} \right)^2 \left( \frac{1}{7} \right)^1 + \left( \begin{array}{c} 7 \\ 2, 1, 4 \end{array} \right) \left( \frac{4}{7} \right)^2 \left( \frac{2}{7} \right)^1 \left( \frac{1}{7} \right)^4 + \\
&\left( \begin{array}{c} 7 \\ 0, 0, 7 \end{array} \right) \left( \frac{4}{7} \right)^0 \left( \frac{2}{7} \right)^0 \left( \frac{1}{7} \right)^7 = \\
&\left( 105 \cdot 4^2 \cdot 1^1 \right) + 105 \left( 4^2 \cdot 2^1 \cdot 1^4 \right) + 1 \right) / 7^7 = 110881 / 7^7.
\end{align*} \]

(d) What is the probability that there are fewer than 3 sunny days in the week?

**Solution:**

\[ P[0 \text{ sunny days}] + P[1 \text{ sunny day}] + P[2 \text{ sunny days}] = \]

\[ \left( \begin{array}{c} 7 \\ 0 \end{array} \right) \left( \frac{1}{7} \right)^0 \left( \frac{6}{7} \right)^7 + \left( \begin{array}{c} 7 \\ 1 \end{array} \right) \left( \frac{1}{7} \right)^1 \left( \frac{6}{7} \right)^6 + \left( \begin{array}{c} 7 \\ 2 \end{array} \right) \left( \frac{1}{7} \right)^2 \left( \frac{6}{7} \right)^5 = \]

\[ \frac{1 + 6^6 + 6^5}{7^7}. \]
10. Now, continuing the previous problem, define a random variable \( C \) that maps outcomes \( s \) in the sample space \( S \) to real numbers according to \( C(s) = \text{number of cloudy days in } s \). (You may assume that it never rains nor snows while the sun is shining.)

(a) What is the sample space \( S \), assuming order matters?

**Solution:** We are now interested only in the event “cloudy” vs. the event “sunny”. From the previous problem we have that \( P[\text{“cloudy”}] = \frac{6}{7} \) and \( P[\text{“sunny”}] = \frac{1}{7} \). The sample space is

\[
S = \{ssssss, sssssc, sssscs, sssssc, sssccc, \ldots, ccccccs, cccccc\}.
\]

(b) What is the range \( S_C \) of \( C \)?

**Solution:** There can be zero, one, two, three, four, five, six, or seven cloudy days in the week so \( S_C = \{0, 1, 2, 3, 4, 5, 6, 7\} \).

(c) What is the PMF \( P_C(c) \)?

**Solution:**

\[
P_C(c) = \begin{cases} 
(\frac{7}{7})(\frac{6}{7})^0(\frac{1}{7})^7 & c = 0 \\
(\frac{7}{7})(\frac{6}{7})^1(\frac{1}{7})^6 & c = 1 \\
(\frac{7}{7})(\frac{6}{7})^2(\frac{1}{7})^5 & c = 2 \\
(\frac{7}{7})(\frac{6}{7})^3(\frac{1}{7})^4 & c = 3 \\
(\frac{7}{7})(\frac{6}{7})^4(\frac{1}{7})^3 & c = 4 \\
(\frac{7}{7})(\frac{6}{7})^5(\frac{1}{7})^2 & c = 5 \\
(\frac{7}{7})(\frac{6}{7})^6(\frac{1}{7})^1 & c = 6 \\
(\frac{7}{7})(\frac{6}{7})^7(\frac{1}{7})^0 & c = 7 
\end{cases}
\]

(d) Define a random variable \( \tilde{C} \) to be the index of the first snowy day \( i.e. \) if it snows on the first day \( \tilde{C} = 1 \), and so forth. Define \( \tilde{C}(s) = 0 \) if there are no snowy days. What is the PMF \( P_{\tilde{C}}(c) \)?

**Solution:**
\[ P_c(c) = \begin{cases} 
\left( \frac{6}{7} \right)^0 \left( \frac{1}{7} \right)^7 & c = 0 \\
\left( \frac{6}{7} \right)^1 \left( \frac{1}{7} \right)^6 & c = 1 \\
\left( \frac{6}{7} \right)^2 \left( \frac{1}{7} \right)^5 & c = 2 \\
\left( \frac{6}{7} \right)^3 \left( \frac{1}{7} \right)^4 & c = 3 \\
\left( \frac{6}{7} \right)^4 \left( \frac{1}{7} \right)^3 & c = 4 \\
\left( \frac{6}{7} \right)^5 \left( \frac{1}{7} \right)^2 & c = 5 \\
\left( \frac{6}{7} \right)^6 \left( \frac{1}{7} \right)^1 & c = 6 \\
\left( \frac{6}{7} \right)^7 \left( \frac{1}{7} \right)^0 & c = 7 
\end{cases} \]