ECE 302 Homework Assignment 7 Partial Solution

Note: To obtain credit for an answer, you must provide adequate justification. Also, if it is possible to obtain a numeric answer, you should not only find an expression for the answer, but also solve for the numeric answer.

1. Consider independent random variables $X$ and $Y$ whose PDF’s are

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

$$f_Y(y) = \begin{cases} 3y^2 & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

and let the event $A = \{X < Y\}$.

Preliminary work: The event $A$ is described by $A = \{(x, y) \mid 0 \leq x \leq y \leq 1\}$. For parts (c) and (d) we will need to find $P[A]$. This is quite straight forward because we are given that $X$ and $Y$ are independent so the joint PDF is just the product of the given marginal PDFs. Thus we have

$$f_{X,Y}(x, y) = \begin{cases} 6xy^2 & 0 \leq x \leq 1, 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

and

$$P[A] = \int_0^1 \int_x^1 6xy^2 \, dy \, dx \quad (3)$$

$$= \int_0^1 6x \left[ y^3 \right]_x^1 \, dx \quad (4)$$

$$= 2 \int_0^1 (x - x^4) \, dx \quad (5)$$

$$= 2 \left( \frac{x^2}{2} - \frac{x^5}{5} \right) \bigg|_0^1 = \frac{3}{5}. \quad (6)$$

We will also need the conditional JPDF $f_{X,Y \mid A}(x, y)$ which, by definition, is

$$f_{X,Y \mid A}(x, y) = \begin{cases} \frac{6xy^2}{3/5} & (x, y) \in A \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $E[X]$.

Solution: Integrating over all values of $x$, we obtain

$$E[X] = \int_0^1 xf_X(x) \, dx = \int_0^1 2x^2 \, dx = \frac{2x^3}{3} \bigg|_0^1 = \frac{2}{3}. $$
(b) Find $E[Y]$.

**Solution:** Integrating over all values of $y$, we obtain

$$E[Y] = \int_0^1 y f_Y(y) \, dy = \int_0^1 3y^3 \, dy = \left. \frac{3y^4}{4} \right|_0^1 = \frac{3}{4}.$$  

(c) Find $E[X|A]$.

**Solution:** Applying Theorem 4.20, we obtain

$$E[X|A] = \int_0^1 \int_x^1 x \left(10xy^2\right) \, dy \, dx = \frac{5}{9}.$$  

(d) Find $E[Y|A]$.

**Solution:** Again applying Theorem 4.20, we obtain

$$E[Y|A] = \int_0^1 \int_x^1 y \left(10xy^2\right) \, dy \, dx = \frac{5}{6}.$$  

2. Suppose that the $i$th photon arrives at a photon detector at time $X_i$ and the $(i+1)$st photon arrives at time $X_{i+1}$ where

$$f_{X_i,X_{i+1}}(x_i,x_{i+1}) = \begin{cases} \lambda^2 e^{-\lambda x_{i+1}} & 0 \leq x_i < x_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

and $\lambda$ is the average number of photons received per second.

(a) Find the conditional PDF $f_{X_{i+1}|X_i}(x_{i+1}|x_i)$.

(b) Find the conditional PDF $f_{X_i|X_{i+1}}(x_i|x_{i+1})$.

(c) Find $E[X_i|X_{i+1}]$.

(d) Find $E[X_{i+1}^2|X_i]$.

(e) Find the moment generating function $\phi_X(s)$ corresponding to each $X_i$.

(f) Find $E[X_i|X_{i+1}]$ using the MGF.
3. Suppose that a discrete-time signal whose values $X_i$ has

$$\text{Cov}[X_i, X_j] = \begin{cases} 
1 & |j - i| = 0 \\
1/4 & |j - i| = 1 \\
0 & \text{otherwise}.
\end{cases}$$

We filter the signal to obtain

$$Y_i = (X_{i-1} + 2X_i + X_{i+1}) / 4.$$

(a) Find $E[Y_i]$.

**Solution:** Linearity of the expected value yields

$$E[Y_i] = E[(X_{i-1} + 2X_i + X_{i+1}) / 4] = \frac{1}{4} E[X_{i-1}] + \frac{1}{2} E[X_i] + \frac{1}{4} E[X_{i+1}].$$

(b) Find $\text{Var}[Y_i]$.

**Solution:** Applying Theorem 3.5(d) and Theorem 6.2 and then using the given information about the covariance we obtain we obtain

$$\text{Var}[Y_i] = \text{Var}\left[\frac{1}{4}X_{i-1}\right] + \text{Var}\left[\frac{1}{2}X_i\right] + \text{Var}\left[\frac{1}{4}X_{i+1}\right] + 2\text{Cov}\left[\frac{1}{4}X_{i-1}, \frac{1}{2}X_i\right] + 2\text{Cov}\left[\frac{1}{4}X_{i-1}, \frac{1}{4}X_{i+1}\right] + 2\text{Cov}\left[\frac{1}{2}X_i, \frac{1}{4}X_{i+1}\right]$$

$$= \frac{1}{16} + \frac{1}{4} + \frac{1}{16} + 2(2)\frac{1}{64} + 2(0) = \frac{7}{16}.$$

4. Suppose that we model the number of data packets arriving per minute at a switch using a Poisson distribution so that $M_i$, the number of packets in the $i$th minute has PMF

$$P_{M_i}(m) = \begin{cases} 
2^m e^{-2}/m! & m \in \{0, 1, 2, \ldots\} \\
0 & \text{otherwise}.
\end{cases}$$

Let $N_i$ be the total number of packets arriving in the first $i$ minutes.

(a) Find $\phi_{M_i}(s)$.

**Solution:** The MGF for a Poisson random variable with parameter 2 is

$$\phi_{M_i}(s) = e^{2(e^s-1)}.$$

(b) Find $\phi_{N_i}(s)$.

**Solution:** The MGF of a sum of random variables is the product of the individual MGF’s so using the fact that the $M_i$ are iid, we obtain

$$\phi_{N_i}(s) = \left(e^{2(e^s-1)}\right)^i = e^{2i(e^s-1)}.$$
(c) Find the PMF of \( N_i \).

**Solution:** The PMF of \( N_i \) is thus that of another Poisson distribution, this one with parameter \( 2i \), so

\[
P_{N_i}(n) = \begin{cases} 
(2i)^ne^{-2i}/n! & i \in \{0, 1, 2, \ldots\} \\
0 & \text{otherwise.}
\end{cases}
\]

(d) Find the expected value of \( N_i \).

**Solution:** Using the properties of the Poisson distribution, \( E[N_i] = 2i \).

(e) Find the variance of \( N_i \).

**Solution:** Using the properties of the Poisson distribution, \( \text{Var}[N_i] = 2i \).

5. Let \( X \) be the height in centimeters of a randomly chosen adult and suppose that \( X \) is normally distributed with mean 165 and standard deviation 30.

**Preliminary work:** Given a constant \( c \) and an arbitrary random variable \( X \), Chernoff’s bound is given by

\[
P[X \geq c] \leq \min_{s \geq 0} e^{-sc} \phi_X(s) =: \min_{s \geq 0} h(s).
\]

Since our \( X \) is normally distributed, \( \phi_X(s) = e^{s\mu + s^2\sigma^2/2} \). Then \( h(s) = e^{s(\mu - c) + s^2\sigma^2/2} \) and

\[
\frac{d}{ds} e^{s(\mu - c) + s^2\sigma^2/2} = (\mu - c + s\sigma^2) e^{s(\mu - c) + s^2\sigma^2/2}.
\]

Setting the right hand side equal to zero yields that an extremum of \( h(s) \) occurs at

\[
s = \left( \frac{c - \mu}{\sigma^2} \right)
\]

(\( e^{\text{blah} } \) is never zero) and checking the second derivative verifies that this extremum is a minimum.

The Chernoff bound is thus \( e^{-\left(\frac{(\mu-c)^2}{2\sigma^2}\right)} \). Substituting for \( \mu \) and \( \sigma \) yields \( e^{-\left(\frac{(165-c)^2}{1800}\right)} \).

We can thus write a Matlab script to generate the answers needed. Here’s such a script:

```matlab
c = [165,215,265,315];
for cindex = 1:4,
    chernoff_bound(cindex) = exp(-(165-c(cindex))^2/1800);
    exact_prob(cindex) = 1-cdf('norm',c(cindex),165,30);
end;
disp(['c = ',int2str(c)])
disp(['Chernoff bound = ',num2str(chernoff_bound)])
disp(['Exact Probability = ',num2str(exact_prob)])
```
(a) Find the exact probability that $X > 165$.

**Solution:** Using the above script yields $P(X > 165) = 1/2$.

(b) Use the Chernoff bound to find an upper bound on that probability.

**Solution:** Using the above script yields $P(X > 165) \leq 1$.

(c) Find the exact probability that $X > 215$.

**Solution:** Using the above script yields $P(X > 215) = 0.048$.

(d) Use the Chernoff bound to find an upper bound on that probability.

**Solution:** Using the above script yields $P(X > 215) \leq 0.25$.

(e) Find the exact probability that $X > 265$.

**Solution:** Using the above script yields $P(X > 265) = 4.3 \times 10^4$.

(f) Use the Chernoff bound to find an upper bound on that probability.

**Solution:** Using the above script yields $P(X > 265) \leq 3.9 \times 10^3$.

(g) Find the exact probability that $X > 315$.

**Solution:** Using the above script yields $P(X > 315) = 2.87 \times 10^7$.

(h) Use the Chernoff bound to find an upper bound on that probability.

**Solution:** Using the above script yields $P(X > 315) \leq 3.72 \times 10^6$. 