1. At time $t = 0$, you begin counting the arrivals of buses at a depot. The number of buses $K_i$ that arrive between time $i - 1$ minutes and time $i$ minutes, has the Poisson PMF

$$P_{K_i}(k) = \begin{cases} 3^k e^{-3}/k! & k = 0, 1, 2, \ldots, \\ 0 & \text{otherwise.} \end{cases}$$

(1)

and $K_1, K_2, \ldots$ are an iid random sequence. Let $R_i = K_1 + K_2 + \cdots + K_i$ denote the number of buses arriving in the first $i$ minutes.

(a) What is the moment generating function $\phi_{K_i}(s)$?

(b) Find the MGF $\phi_{R_i}(s)$.

(c) Find the PMF $P_{R_i}(r)$. Hint: Compare $\phi_{R_i}(s)$ and $\phi_{K_i}(s)$.

(d) Find $E[R_i]$ and $\text{Var}[R_i]$.

(e) Let $N$ be a geometric random variable with PMF

$$P_N(n) = \begin{cases} (1 - q)q^{n-1} & n = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(2)

Find the MGF and PDF of $R_N$.

(a) Poisson (3) $\Rightarrow \phi_{K_i}(s) = e^{3(e^s - 1)}$

(b) MGF of sum is product of MGFs $\Rightarrow \phi_{R_i}(s) = (e^{3(e^s - 1)})^i = e^{3i(e^s - 1)}$

(c) PMF of r.v. $R_i$ is thus

$$P_{R_i}(r) = \begin{cases} (3i)^r e^{-3i} / r! & r = 0, 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases}$$

(d) $E[R_i] = \left[ \frac{d}{ds} \phi_{K_i}(s) \right]_{s=0} = 3i e^{3(e^s - 1)} e^s \bigg|_{s=0} = 3i$

Let $N = e^s - 1$ then $dN = e^s$ and use chain rule.
3. In an automatic geolocation system, a dispatcher sends a message to six trucks in a fleet asking for their locations. The waiting times for responses from the six trucks are iid exponential random variables, each with expected value 2 seconds.

(a) What is the probability that all six responses will arrive within 5 seconds?

(b) (Bonus Question): If the system has to locate all six vehicles within 3 seconds, it has to reduce the expected response time of each vehicle. What is the maximum expected response time that will produce a location time for all six vehicles of 3 seconds or less with probability of at least 0.9?

\[
\begin{align*}
\text{a)} & \quad P \left[ x_i < 5 \quad \forall i \in \{1, 2, 3, 4, 5, 6\} \right] = \prod_{i=1}^{6} P \left[ x_i < 5 \right] = \\
& \left( \int_{0}^{5} \lambda e^{-\lambda x} \, dx \right)^6 \quad \text{where} \quad \lambda = \frac{1}{E[x]} = \frac{1}{2} \\
& \therefore P[\text{ALL SIX RESPONSES ARRIVE WITHIN 5 sec}] = \\
& \left[ \int_{0}^{5} \lambda e^{-\lambda x} \, dx \right]^6 = \left[ -e^{-\lambda x} \bigg|_{0}^{5} \right]^6 = \left( 1 - e^{-5\lambda} \right)^6 = \left( 1 - e^{-\frac{5}{2}} \right)^6 \\
\text{b)} & \quad P \left[ x_i < 3 \quad \forall i \in \{1, 2, 3, 4, 5, 6\} \right] \geq 0.9 \quad \text{REQUIRES} \quad (1 - e^{-3\lambda}) \geq 0.9 \quad \text{SO NEED} \quad (1 - e^{-3\lambda}) \geq (0.9)^\frac{1}{3} \\
& \text{i.e.,} \quad 1 - \frac{\sqrt[3]{0.9}}{3} \geq e^{-3\lambda} \quad \text{i.e.,} \quad \ln \left( 1 - \frac{\sqrt[3]{0.9}}{3} \right) \geq -3\lambda \quad \text{i.e.,} \quad \lambda \geq -\frac{\ln \left( 1 - \frac{\sqrt[3]{0.9}}{3} \right)}{3} \quad \text{E[x]} \leq \frac{1}{\lambda} \geq \frac{1}{\frac{\ln \left( 1 - \frac{\sqrt[3]{0.9}}{3} \right)}{3}} \\

\text{1d cont.} \quad E[R_i^2] = \left[ \frac{d^2}{ds^2} \Phi_R(s) \right] = \left[ \frac{d}{ds} \left. 3i e^{3i(e^s-1)} e^s \right|_{s=0} \right] = \left[ 3i (3i e^{3i(e^s-1)}) e^s + 3i e^{3i(e^s-1)} e^s \right]_{s=0} = 3i e^{3i(e^0-1)} (3i e^0 + 1) |_{s=0} = 3i (3i + 1) \\
& \text{Thus} \quad \text{VAR}[R_i] = \frac{3i (3i + 1) - (3i)^2}{E[R_i^2]} = \frac{3i}{E[R_i^2]} \quad \text{AS EXPECTED} \quad \text{SINCE } R_i \text{ IS Poisson} \\
\text{c)} & \quad \Phi_R(s) = \frac{(1-q)e^s}{1-ge^s} \quad \text{so} \quad \frac{d}{ds} \Phi_R(s) = \frac{(1-q)e^s}{(1-ge^s)^2} \quad \text{POISSON} \\
\text{WHICH IS THE MGF OF A RASCAL R.V. WITH PARAMETERS } k, \frac{1}{1-q}, \text{ SO } P_R(k, n) = \left( 1 - q \right)^{k-n} q^{n-k}
\end{align*}
\]
4. Random variables $X$ and $Y$ have the joint CDF

$$F_{X,Y}(x,y) = \begin{cases} 
(1 - e^{-2x})(1 - e^{-y}) & x \geq 0; \\
0 & y \geq 0,
\end{cases} \quad (3)$$

(a) What is $P[X \leq 2, Y \leq 3]$?
(b) What is the marginal CDF, $F_X(x)$?
(c) What is the marginal CDF, $F_Y(y)$?
(d) Are $X$ and $Y$ independent? (Justify your answer.)

(a) $P[X \leq 2, Y \leq 3] = F_{X,Y}(2,3) = (1 - e^{-4})(1 - e^{-3})$

(b) $F_{X,Y}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{X,Y}(x,y) = \frac{\partial}{\partial x} (1 - e^{-2x}) e^{-y} = 2e^{-2x} e^{-y}$

so $F_X(x) = \int_0^x 2e^{-2u} e^{-y} dy = -2e^{-2x} e^{-y}\bigg|_0^\infty = 2e^{-2x}$

Thus $F_X(x) = \int_0^x 2e^{-2u} du = -e^{-2u}\bigg|_0^x = 1 - e^{-2x}$

(c) $f_Y(y) = \int_0^\infty 2e^{-2x} e^{-y} dx = -e^{-2x} e^{-y}\bigg|_0^\infty = e^{-y}$

so $F_Y(y) = \int_0^y e^{-v} dv = -e^{-v}\bigg|_0^y = 1 - e^{-y}$

(d) $f_{X,Y}(x,y) = 2e^{-2x} e^{-y} = f_X(x) f_Y(y)$ so $X$ and $Y$ are independent.
5. Random variables $X$ and $Y$ have the joint PMF

$$P_{X,Y}(x,y) = \begin{cases} \ c_{xy} & x = 1, 2, 3; \ y = 1, 2, \\ 0 & \text{otherwise}. \end{cases} \quad (4)$$

(a) What is the value of the constant $c$?
(b) What is $P[Y < X]$?
(c) What is $P[Y = X]$?
(d) What is $P[Y > X]$?
(e) What is $P[Y = 2]$?
(f) What are the marginal PMFs $P_X(x)$ and $P_Y(y)$?
(g) What are the expected values $E[X]$ and $E[Y]$?
(h) What are the standard deviations $\sigma_X$ and $\sigma_Y$?
(i) If $W = X - Y$, find the probability mass function $P_W(w)$,
(j) Find the expected value $E[W]$,
(k) Find the $P[W > 0]$.
(l) Find the correlation, $E[XY]$.
(m) Find the covariance, Cov $[X,Y]$,
(n) Find the correlation coefficient, $\rho_{X,Y}$,
(o) Find the variance of $X + Y$, Var $[X + Y]$. 

(a) \[ \sum_{x=1}^{3} \sum_{y=1}^{2} c_{xy} = c + 2c + 2c + 4c + 3c + 6c = 18c = 1 \]
so \[ c = \frac{1}{18} \]

(b) \[ P[Y < X] = P[(1,1)] + P[(3,1)] + P[(3,2)] \]
\[ = \frac{1}{18} \left[ 2 + 3 + 6 \right] = \frac{11}{18} \]

(c) \[ P[Y = X] = P[(1,1)] + P[(2,2)] = \frac{5}{18} \]

(d) \[ P[Y > X] = P[(2,1)] = \frac{2}{18} = \frac{1}{9} \]

(e) \[ P[Y = 2] = P[(1,2)] + P[(2,2)] + P[(3,2)] = \frac{12}{18} = \frac{2}{3} \]

(f) \[ P_X(x) = \frac{1}{18} \sum_{y=1}^{2} x_y = \frac{3}{18} = \frac{x}{6} \]
\[ P_Y(y) = \frac{1}{18} \sum_{x=1}^{3} x_y = \frac{3}{18} = \frac{y}{6} \]
\[ E[X] = \sum_{x=1}^{3} x P_x(x) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} = 14/6 = 7/3 \]
\[ E[Y] = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} = 5/3 \]

(g) \[ E[XY] = \sum_{x=1}^{3} \sum_{y=1}^{2} x y P_{X,Y}(x,y) = \frac{1}{6} + \frac{2}{6} + \frac{3}{6} = 14/6 = 7/3 \]
(n) \[ \sigma_x = \sqrt{\text{Var}(X)} = \sqrt{E[X^2] - (E[X])^2} \quad \text{so } E[X^2] = \frac{1}{6} + \frac{2}{6} + \frac{2}{6} = \frac{3}{2} = \sigma_x \]

\[ \sigma_y = \sqrt{\frac{2}{3} - \left(\frac{2}{3}\right)^2} = \frac{\sqrt{2}}{3} \]

(i) \[ W = X - Y \quad \text{F} \equiv \mathbb{O} \quad P_w(w) \]

\[ S_w = \{0, 1, 2, 3, 4\} \quad \text{so } P_w(w) = \begin{cases} \frac{1}{18} & w = 0 \\ \frac{1}{18} & w = -1 \\ \frac{1}{18} & w = 1 \\ \frac{1}{18} & w = 2 \\ \frac{1}{18} & w = 3 \\ \frac{1}{18} & w = 4 \end{cases} \]

CHECK: \[ \frac{1}{18} (0 + 2 + 8 + 3) = 1 \]

(j) \[ E[W] = \frac{5}{18} (0) + \frac{2}{18} (-1) + \frac{2}{18} (1) + \frac{3}{18} (2) = \frac{12}{18} = \frac{2}{3} \]

(k) \[ P(W > 0) = 1 - P(W \leq 0) = 1 - \frac{4}{9} = \frac{5}{9} \]

(l) \[ E[XY] = \sum_{x=1}^{3} \sum_{y=1}^{2} xy P_{x,y}(x,y) = \frac{25}{18} \sum_{x=1}^{3} \sum_{y=1}^{2} x^2 y^2 \]

\[ = \frac{1}{18} (1 + 4 + 4 + 16 + 36 + 10) = \frac{71}{18} \]

(m) \[ \text{Cov}[X,Y] = E[(X-\mu_x)(Y-\mu_y)] = E[XY] - E[X]E[Y] = \frac{71}{18} - \frac{35}{9} = \frac{1}{9} \]

(n) \[ \rho_{xy} = \frac{\text{Cov}[X,Y]}{\sigma_x \sigma_y} = \frac{\frac{1}{18}}{\left(\frac{\sqrt{2}}{3}\right)^2} = \frac{9}{18} = \frac{1}{18} \]

(o) \[ \text{Var}[X+Y] = E[(X+Y)^2] - E[(X+Y)]^2 \]

\[ E[X+Y] = \sum_{x=1}^{3} \sum_{y=1}^{2} (x+y) P_{x,y}(x,y) = \frac{25}{18} \sum_{x=1}^{3} x P_{x,y}(x,y) + \frac{25}{18} \sum_{y=1}^{2} y P_{x,y}(x,y) \]

\[ = \frac{3}{18} \sum_{x=1}^{3} x^2 y + \frac{1}{18} \sum_{x=1}^{3} x y^2 \]

\[ = \frac{1}{18} \left( \frac{\sum_{x=1}^{3} (x+1)}{1+2+4+2+4} \right) = 72/18 = 36/9 = 4 \]


\[ = 6 + 2 \left( \frac{35}{9} \right) + 3 = 9 + 80/9 = 16 + 7/9 \]
6. X and Y are independent random variables with PDFs

\[ f_X(x) = \begin{cases} \frac{1}{3} e^{-x/3} & x \geq 0, \\ 0 & \text{otherwise,} \end{cases} \quad (5) \]

\[ f_Y(y) = \begin{cases} \frac{1}{2} e^{-y/2} & y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \quad (6) \]

(a) What is the joint PDF \( f_{X,Y}(x, y) \)?

(b) What is \( P[X > Y] \)?

(c) What is \( E[XY] \)?

(d) What is \( \text{Cov}[XY] \)?

(a) \( X, Y \text{ i.i.d.} \Rightarrow f_{X,Y}(x, y) = f_X(x) f_Y(y) = \begin{cases} \frac{1}{6} e^{-x/3} e^{-y/2} & x, y \geq 0, \\ 0 & \text{otherwise.} \end{cases} \)

(b) \( P[X > Y] = \int_0^\infty \int_y^\infty \frac{1}{6} e^{-x/3} e^{-y/2} \, dx \, dy \)

\[ = -\left[ \frac{1}{2} e^{-x/3} \right]_y^\infty e^{-y/2} \, dy = \int_0^\infty \frac{1}{2} e^{-5y/6} \, dy \]

\[ = -\frac{1}{2} (\frac{6}{5}) e^{-5y/6} \bigg|_0^\infty = \frac{1}{2} (\frac{6}{5}) = \frac{3}{5} \]

(c) \( E[XY] = E[X]E[Y] = \frac{3 \cdot 2}{6} = 6 \), \( E[X] \) \& \( E[Y] \) both exponential

(d) \( \text{Cov}[X, Y] = 0 \)

\( \frac{5}{6} \text{ cont.} \)

so \( \text{Var}[X + Y] = 16 + \frac{1}{9} - 16 = \frac{1}{9} \)
2. The random variable $Y$ has expected value $\mu_Y = 3$. We take 4 independent samples of the random variable $Y$. Chebyshev’s inequality states that $P\left[ \frac{1}{4} \leq M_4(Y) \leq 5 \right] \geq 0.5$.

(a) What is $E[M_4(Y)]$?
(b) What is the variance of $M_4(Y)$?
(c) What is the variance of $Y$?

(a) $E[M_4(Y)] = E[Y]$ because the sample mean is an unbiased estimator.

(b) $1 < M_4(Y) < 5 \iff -2 < M_4(Y) - E[M_4(Y)] < 2$ so $P\left[ \left| M_4(Y) - 3 \right| < 2 \right] \geq 0.5$ is given.

$$P\left[ \left| M_4(Y) - 3 \right| < 2 \right] \geq 0.5 \Rightarrow P\left[ \left| M_4(Y) - 3 \right| \geq 2 \right] \leq 1 - 0.5$$

So

$$\frac{\text{Var}[M_4(Y)]}{2^2} = \frac{1}{2}$$

$$\Rightarrow \text{Var}[M_4(Y)] = 2$$

(c) $\text{Var}[M_4(Y)] = \frac{\text{Var}[Y]}{4}$ so $\text{Var}[Y] = 8$.
3. The number of calls received at a telephone switch has a Poisson distribution with rate 5 per second on weekdays (Monday through Friday) and 2 per second on weekends (Saturday and Sunday). Let $N$ be the number of calls received between 7:01:01 and 7:01:02 a.m. on a particular day. Given a value for $N$ on a particular day, you would like to test the hypothesis $H_0$ that this day is a weekday.

(a) Suppose that you decide that the day was a weekday if $N \geq 3$. What is the probability of mistakenly classifying a weekend day as a weekday?

(b) Using this same threshold, what is the probability of mistakenly classifying a weekday as a weekend day?

(c) What is the probability of misclassifying the day for which you are given a value of $N$?

(a) If we decide that the day is a weekday whenever $N \geq 3$, the prob. that we decide weekend when weekend is true is just $P[N \geq 3 | \lambda = 2] = 1 - P[N \leq 2 | \lambda = 2]$

\[
P[N \leq 2 | \lambda = 2] = \sum_{n=0}^{2} \frac{2^n e^{-2}}{n!} = e^{-2} (1 + 1 + 2) = 4e^{-2}
\]

(b) Otoh, the prob. that we decide weekend when weekday is $P[N \leq 2 | \lambda = 5] = \sum_{n=0}^{2} \frac{5^n e^{-5}}{n!} = e^{-5} (1 + 5 + \frac{25}{2}) = (37) e^{-5}$

(c) The total prob. of error is thus $1 - 4e^{-2} + \frac{37}{2} e^{-5}$
5. (Bonus Question) The duration of an ECE302 exam is a Gaussian random variable with expected value 3 hours and variance 1 hour. All exams have independent durations. In a three-year period someone takes ECE302 five times. The total time this lucky person spends taking ECE302 exams during the three-year period is a random variable $T$ (in hours).

(a) What is the expected value $\mu_T$?
(b) What is the variance of $T$?
(c) Use the central limit theorem to estimate the probability that the student spends more than 24 hours taking ECE302 exams over the five year period. (Table 3.1 is on page 123 of your textbook.)

$$D \sim \mathcal{N}(3, 1)$$

**LET** $$T = \sum_{i=1}^{5} D_i$$

(a) $$E[T] = 5E[D] = 15$$

(b) $$\text{Var}[T] = \sum_{i=1}^{5} \text{Var}[D_i] = 5(1)$$

**IND. GAUS$\Rightarrow$UNCORR.$**

(c) $$Z = \frac{T - E[T]}{\sqrt{\text{Var}[T]}} = \frac{T - 15}{\sqrt{5}}$$

so $$P[T > 24] = 1 - P[T \leq 24]$$

$$= 1 - \Phi \left( \frac{24 - 15}{\sqrt{5}} \right) = 1 - \Phi \left( \frac{9}{\sqrt{5}} \right)$$

(*OUT IRRELEVANT SINCE ALREADY HAVE GAUSSIAN R.V.*)