Solutions to HW3

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate. I have also largely rewritten the solutions to problems 2.10.1, 2.10.2, and 2.10.3.

Problem 2.2.1 •

The random variable N has PMF

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0, 1, 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[N \leq 1]$?

Problem 2.2.1 Solution

(a) We wish to find the value of c that makes the PMF sum up to one.

$$P_N(n) = \begin{cases} c(1/2)^n & n = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Therefore, $\sum_{n=0}^{2} P_N(n) = c + c/2 + c/4 = 1$, implying $c = 4/7$.

(b) The probability that $N \leq 1$ is

$$P[N \leq 1] = P[N = 0] + P[N = 1] = 4/7 + 2/7 = 6/7 \tag{2}$$

Problem 2.2.2 •

For random variables X and R defined in Example 2.5, find $P_X(x)$ and $P_R(r)$. In addition, find the following probabilities:

(a) $P[X = 0]$

(b) $P[X < 3]$

(c) $P[R > 1]$

Problem 2.2.2 Solution

From Example 2.5, we can write the PMF of X and the PMF of R as

$$P_X(x) = \begin{cases} 1/8 & x = 0 \\ 3/8 & x = 1 \\ 3/8 & x = 2 \\ 1/8 & x = 3 \\ 0 & \text{otherwise} \end{cases} \qquad P_R(r) = \begin{cases} 1/4 & r = 0 \\ 3/4 & r = 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

From the PMFs $P_X(x)$ and $P_R(r)$, we can calculate the requested probabilities

(a) $P[X = 0] = P_X(0) = 1/8$.

(b) $P[X < 3] = P_X(0) + P_X(1) + P_X(2) = 7/8$.

(c) $P[R > 1] = P_R(2) = 3/4$.

Problem 2.2.3 •

The random variable V has PMF

$$P_V(v) = \begin{cases} cv^2 & v = 1, 2, 3, 4, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of the constant c.

(b) Find $P[V \in \{u^2 | u = 1, 2, 3, \cdots\}]$.

(c) Find the probability that V is an even number.

(d) Find $P[V > 2]$.

Problem 2.2.3 Solution

(a) We must choose c to make the PMF of V sum to one.

$$\sum_{v=1}^{4} P_V(v) = c(1^2 + 2^2 + 3^2 + 4^2) = 30c = 1 \tag{1}$$

Hence $c = 1/30$.

(b) Let $U = \{u^2 | u = 1, 2, \ldots\}$ so that

$$P[V \in U] = P_V(1) + P_V(4) = \frac{1}{30} + \frac{4^2}{30} = \frac{17}{30} \tag{2}$$

(c) The probability that V is even is

$$P[V \text{ is even}] = P_V(2) + P_V(4) = \frac{2^2}{30} + \frac{4^2}{30} = \frac{2}{3} \tag{3}$$

(d) The probability that $V > 2$ is

$$P[V > 2] = P_V(3) + P_V(4) = \frac{3^2}{30} + \frac{4^2}{30} = \frac{5}{6} \tag{4}$$

Problem 2.2.4 •
The random variable X has PMF

$$P_X(x) = \begin{cases} c/x & x = 2, 4, 8, \\ 0 & \text{otherwise.} \end{cases}$$

(a) What is the value of the constant c?

(b) What is $P[X = 4]$?

(c) What is $P[X < 4]$?

(d) What is $P[3 \leq X \leq 9]$?

Problem 2.2.4 Solution

(a) We choose c so that the PMF sums to one.

$$\sum_x P_X(x) = \frac{c}{2} + \frac{c}{4} + \frac{c}{8} = \frac{7c}{8} = 1 \tag{1}$$

Thus $c = 8/7$.

(b)

$$P[X = 4] = P_X(4) = \frac{8}{7 \cdot 4} = \frac{2}{7} \tag{2}$$

(c)

$$P[X < 4] = P_X(2) = \frac{8}{7 \cdot 2} = \frac{4}{7} \tag{3}$$

(d)

$$P[3 \leq X \leq 9] = P_X(4) + P_X(8) = \frac{8}{7 \cdot 4} + \frac{8}{7 \cdot 8} = \frac{3}{7} \tag{4}$$

Problem 2.3.1 •
In a package of M&Ms, Y, the number of yellow M&Ms, is uniformly distributed between 5 and 15.

(a) What is the PMF of Y?

(b) What is $P[Y < 10]$?

(c) What is $P[Y > 12]$?

(d) What is $P[8 \leq Y \leq 12]$?

Problem 2.3.1 Solution

(a) If it is indeed true that Y, the number of yellow M&M's in a package, is uniformly distributed between 5 and 15, then the PMF of Y, is

$$P_Y(y) = \begin{cases} 1/11 & y = 5, 6, 7, \ldots, 15 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b)

$$P[Y < 10] = P_Y(5) + P_Y(6) + \cdots + P_Y(9) = 5/11 \tag{2}$$

(c)

$$P[Y > 12] = P_Y(13) + P_Y(14) + P_Y(15) = 3/11 \tag{3}$$

(d)

$$P[8 \le Y \le 12] = P_Y(8) + P_Y(9) + \cdots + P_Y(12) = 5/11 \tag{4}$$

Problem 2.3.4 •

Anytime a child throws a Frisbee, the child's dog catches the Frisbee with probability p, independent of whether the Frisbee is caught on any previous throw. When the dog catches the Frisbee, it runs away with the Frisbee, never to be seen again. The child continues to throw the Frisbee until the dog catches it. Let X denote the number of times the Frisbee is thrown.

(a) What is the PMF $P_X(x)$?

(b) If $p = 0.2$, what is the probability that the child will throw the Frisbee more than four times?

Problem 2.3.4 Solution

(a) Let X be the number of times the frisbee is thrown until the dog catches it and runs away. Each throw of the frisbee can be viewed as a Bernoulli trial in which a success occurs if the dog catches the frisbee an runs away. Thus, the experiment ends on the first success and X has the geometric PMF

$$P_X(x) = \begin{cases} (1-p)^{x-1}p & x = 1, 2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The child will throw the frisbee more than four times iff there are failures on the first 4 trials which has probability $(1-p)^4$. If $p = 0.2$, the probability of more than four throws is $(0.8)^4 = 0.4096$.

Note: There is a less elegant but equally effective way to solve the problem, which I show below is equivalent.

$$\begin{aligned} P[X > 4] &= 1 - (P[X = 4] + P[X = 3] + P[X = 2] + P[X = 1]) & (2)\\ &= 1 - \left(p(1-p)^3 + p(1-p)^2 + p(1-p) + p\right) & (3)\\ &= 1 - 4p + 6p^2 - 4p^3 + p^4 & (4)\\ &= (1-p)^4 & (5) \end{aligned}$$

Problem 2.3.6 •

The number of bits B in a fax transmission is a geometric ($p = 2.5 \cdot 10^{-5}$) random variable. What is the probability $P[B > 500{,}000]$ that a fax has over 500,000 bits?

Problem 2.3.6 Solution

The probability of more than 500,000 bits is

$$\begin{aligned} P[B > 500{,}000] &= 1 - \sum_{b=1}^{500{,}000} P_B(b) & (1)\\ &= 1 - p \sum_{b=1}^{500{,}000} (1-p)^{b-1} & (2) \end{aligned}$$

Math Fact B.4 implies that $(1-x)\sum_{b=1}^{500{,}000} x^{b-1} = 1 - x^{500{,}000}$. Substituting, $x = 1 - p$, we obtain:

$$\begin{aligned} P[B > 500{,}000] &= 1 - (1 - (1-p)^{500{,}000}) & (3)\\ &= (1 - 2.5 \times 10^{-5})^{500{,}000} \approx 0.3726 \times 10^{-5} & (4) \end{aligned}$$

Problem 2.3.7 •

The number of buses that arrive at a bus stop in T minutes is a Poisson random variable B with expected value $T/5$.

(a) What is the PMF of B, the number of buses that arrive in T minutes?

(b) What is the probability that in a two-minute interval, three buses will arrive?

(c) What is the probability of no buses arriving in a 10-minute interval?

(d) How much time should you allow so that with probability 0.99 at least one bus arrives?

Problem 2.3.7 Solution

Since an average of $T/5$ buses arrive in an interval of T minutes, buses arrive at the bus stop at a rate of $1/5$ buses per minute.

(a) From the definition of the Poisson PMF, the PMF of B, the number of buses in T minutes, is

$$P_B(b) = \begin{cases} (T/5)^b e^{-T/5}/b! & b = 0, 1, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) Choosing $T = 2$ minutes, the probability that three buses arrive in a two minute interval is

$$P_B(3) = (2/5)^3 e^{-2/5}/3! \approx 0.0072 \tag{2}$$

(c) By choosing $T = 10$ minutes, the probability of zero buses arriving in a ten minute interval is

$$P_B(0) = e^{-10/5}/0! = e^{-2} \approx 0.135 \tag{3}$$

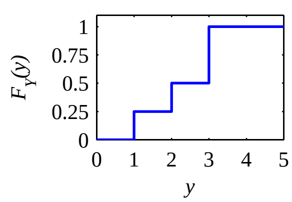
(d) The probability that at least one bus arrives in T minutes is

$$P[B \geq 1] = 1 - P[B = 0] = 1 - e^{-T/5} \geq 0.99 \tag{4}$$

Rearranging yields $T \geq 5 \ln 100 \approx 23.0$ minutes.

Problem 2.4.1 •

Discrete random variable Y has the CDF $F_Y(y)$ as shown:



Use the CDF to find the following probabilities:

(a) $P[Y < 1]$

(b) $P[Y \leq 1]$

(c) $P[Y > 2]$

(d) $P[Y \geq 2]$

(e) $P[Y = 1]$

(f) $P[Y = 3]$

(g) $P_Y(y)$

Problem 2.4.1 Solution

Using the CDF given in the problem statement we find that

(a) $P[Y < 1] = 0$

(b) $P[Y \leq 1] = 1/4$

(c) $P[Y > 2] = 1 - P[Y \leq 2] = 1 - 1/2 = 1/2$

(d) $P[Y \geq 2] = 1 - P[Y < 2] = 1 - 1/4 = 3/4$

(e) $P[Y = 1] = 1/4$

(f) $P[Y = 3] = 1/2$

(g) From the staircase CDF of Problem 2.4.1, we see that Y is a discrete random variable. The jumps in the CDF occur at at the values that Y can take on. The height of each jump equals the probability of that value. The PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 2.4.3 •

The random variable X has CDF

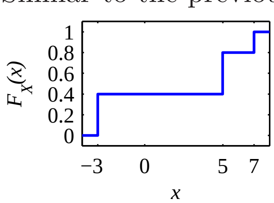
$$F_X(x) = \begin{cases} 0 & x < -3, \\ 0.4 & -3 \leq x < 5, \\ 0.8 & 5 \leq x < 7, \\ 1 & x \geq 7. \end{cases}$$

(a) Draw a graph of the CDF.

(b) Write $P_X(x)$, the PMF of X.

Problem 2.4.3 Solution

(a) Similar to the previous problem, the graph of the CDF is shown below.



$$F_X(x) = \begin{cases} 0 & x < -3 \\ 0.4 & -3 \leq x < 5 \\ 0.8 & 5 \leq x < 7 \\ 1 & x \geq 7 \end{cases} \tag{1}$$

(b) The corresponding PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

Problem 2.5.2 •
Voice calls cost 20 cents each and data calls cost 30 cents each. C is the cost of one telephone call. The probability that a call is a voice call is $P[V] = 0.6$. The probability of a data call is $P[D] = 0.4$.

(a) Find $P_C(c)$, the PMF of C.

(b) What is $E[C]$, the expected value of C?

Problem 2.5.2 Solution
Voice calls and data calls each cost 20 cents and 30 cents respectively. Furthermore the respective probabilities of each type of call are 0.6 and 0.4.

(a) Since each call is either a voice or data call, the cost of one call can only take the two values associated with the cost of each type of call. Therefore the PMF of X is

$$P_X(x) = \begin{cases} 0.6 & x = 20 \\ 0.4 & x = 30 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(b) The expected cost, $E[C]$, is simply the sum of the cost of each type of call multiplied by the probability of such a call occurring.

$$E[C] = 20(0.6) + 30(0.4) = 24 \text{ cents} \tag{2}$$

Problem 2.5.7 •
Find the expected value of a binomial $(n = 5, p = 1/2)$ random variable X.

Problem 2.5.7 Solution
From Definition 2.7, random variable X has PMF

$$P_X(x) = \begin{cases} \binom{5}{x}(1/2)^5 & x = 0, 1, 2, 3, 4, 5 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The expected value of X is

$$\begin{aligned} E[X] &= \sum_{x=0}^{5} x P_X(x) && (2) \\ &= 0\binom{5}{0}\frac{1}{2^5} + 1\binom{5}{1}\frac{1}{2^5} + 2\binom{5}{2}\frac{1}{2^5} + 3\binom{5}{3}\frac{1}{2^5} + 4\binom{5}{4}\frac{1}{2^5} + 5\binom{5}{5}\frac{1}{2^5} && (3) \\ &= [5 + 20 + 30 + 20 + 5]/2^5 = 2.5 && (4) \end{aligned}$$

Problem 2.6.1 •
Given the random variable Y in Problem 2.4.1, let $U = g(Y) = Y^2$.

(a) Find $P_U(u)$.

(b) Find $F_U(u)$.

(c) Find $E[U]$.

Problem 2.6.1 Solution
From the solution to Problem 2.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) Since Y has range $S_Y = \{1, 2, 3\}$, the range of $U = Y^2$ is $S_U = \{1, 4, 9\}$. The PMF of U can be found by observing that

$$P[U = u] = P\left[Y^2 = u\right] = P\left[Y = \sqrt{u}\right] + P\left[Y = -\sqrt{u}\right] \tag{2}$$

Since Y is never negative, $P_U(u) = P_Y(\sqrt{u})$. Hence,

$$P_U(1) = P_Y(1) = 1/4 \qquad P_U(4) = P_Y(2) = 1/4 \qquad P_U(9) = P_Y(3) = 1/2 \tag{3}$$

For all other values of u, $P_U(u) = 0$. The complete expression for the PMF of U is

$$P_U(u) = \begin{cases} 1/4 & u = 1 \\ 1/4 & u = 4 \\ 1/2 & u = 9 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) From the PMF, it is straighforward to write down the CDF.

$$F_U(u) = \begin{cases} 0 & u < 1 \\ 1/4 & 1 \le u < 4 \\ 1/2 & 4 \le u < 9 \\ 1 & u \ge 9 \end{cases} \tag{5}$$

(c) From Definition 2.14, the expected value of U is

$$E[U] = \sum_u u P_U(u) = 1(1/4) + 4(1/4) + 9(1/2) = 5.75 \tag{6}$$

From Theorem 2.10, we can calculate the expected value of U as

$$E[U] = E\left[Y^2\right] = \sum_y y^2 P_Y(y) = 1^2(1/4) + 2^2(1/4) + 3^2(1/2) = 5.75 \tag{7}$$

As we expect, both methods yield the same answer.

Problem 2.6.3 •
Given the random variable X in Problem 2.4.3, let $W = g(X) = -X$.

(a) Find $P_W(w)$.

(b) Find $F_W(w)$.

(c) Find $E[W]$.

Problem 2.6.3 Solution
From the solution to Problem 2.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) The PMF of $W = -X$ satisfies

$$P_W(w) = P[-X = w] = P_X(-w) \tag{2}$$

This implies

$$P_W(-7) = P_X(7) = 0.2 \qquad P_W(-5) = P_X(5) = 0.4 \qquad P_W(3) = P_X(-3) = 0.4 \tag{3}$$

The complete PMF for W is

$$P_W(w) = \begin{cases} 0.2 & w = -7 \\ 0.4 & w = -5 \\ 0.4 & w = 3 \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

(b) From the PMF, the CDF of W is

$$F_W(w) = \begin{cases} 0 & w < -7 \\ 0.2 & -7 \le w < -5 \\ 0.6 & -5 \le w < 3 \\ 1 & w \ge 3 \end{cases} \tag{5}$$

(c) From the PMF, W has expected value

$$E[W] = \sum_w w P_W(w) = -7(0.2) + -5(0.4) + 3(0.4) = -2.2 \tag{6}$$

Problem 2.6.4 •

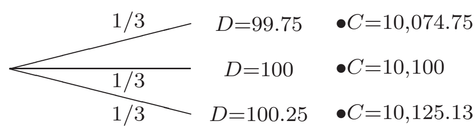
At a discount brokerage, a stock purchase or sale worth less than \$10,000 incurs a brokerage fee of 1% of the value of the transaction. A transaction worth more than \$10,000 incurs a fee of \$100 plus 0.5% of the amount exceeding \$10,000. Note that for a fraction of a cent, the brokerage always charges the customer a full penny. You wish to buy 100 shares of a stock whose price D in dollars has PMF

$$P_D(d) = \begin{cases} 1/3 & d = 99.75, 100, 100.25, \\ 0 & \text{otherwise.} \end{cases}$$

What is the PMF of C, the cost of buying the stock (including the brokerage fee).

Problem 2.6.4 Solution

A tree for the experiment is



Thus C has three equally likely outcomes. The PMF of C is

$$P_C(c) = \begin{cases} 1/3 & c = 10{,}074.75, \quad 10{,}100, \quad 10{,}125.13 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Problem 2.7.1 •

For random variable T in Quiz 2.6, first find the expected value $E[T]$ using Theorem 2.10. Next, find $E[T]$ using Definition 2.14.

Problem 2.7.1 Solution

From the solution to Quiz 2.6, we found that $T = 120 - 15N$. By Theorem 2.10,

$$E[T] = \sum_{n \in S_N} (120 - 15n) P_N(n) \tag{1}$$

$$= 0.1(120) + 0.3(120 - 15) + 0.3(120 - 30) + 0.3(120 - 45) = 93 \tag{2}$$

Also from the solution to Quiz 2.6, we found that

$$P_T(t) = \begin{cases} 0.3 & t = 75, 90, 105 \\ 0.1 & t = 120 \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

Using Definition 2.14,

$$E[T] = \sum_{t \in S_T} t P_T(t) = 0.3(75) + 0.3(90) + 0.3(105) + 0.1(120) = 93 \tag{4}$$

As expected, the two calculations give the same answer.

Problem 2.7.2 •

In a certain lottery game, the chance of getting a winning ticket is exactly one in a thousand. Suppose a person buys one ticket each day (except on the leap year day February 29) over a period of fifty years. What is the expected number $E[T]$ of winning tickets in fifty years? If each winning ticket is worth \$1000, what is the expected amount $E[R]$ collected on these winning tickets? Lastly, if each ticket costs \$2, what is your expected net profit $E[Q]$?

Problem 2.7.2 Solution

Whether a lottery ticket is a winner is a Bernoulli trial with a success probability of 0.001. If we buy one every day for 50 years for a total of $50 \cdot 365 = 18250$ tickets, then the number of winning tickets T is a binomial random variable with mean

$$E[T] = 18250(0.001) = 18.25 \tag{1}$$

Since each winning ticket grosses \$1000, the revenue we collect over 50 years is $R = 1000T$ dollars. The expected revenue is

$$E[R] = 1000E[T] = 18250 \tag{2}$$

But buying a lottery ticket everyday for 50 years, at \$2.00 a pop isn't cheap and will cost us a total of $18250 \cdot 2 = \$36500$. Our net profit is then $Q = R - 36500$ and the result of our loyal 50 year patronage of the lottery system, is disappointing expected loss of

$$E[Q] = E[R] - 36500 = -18250 \tag{3}$$

Problem 2.7.4 •

It can take up to four days after you call for service to get your computer repaired. The computer company charges for repairs according to how long you have to wait. The number of days D until the service technician arrives and the service charge C, in dollars, are described by

$$P_D(d) = \begin{cases} 0.2 & d = 1, \\ 0.4 & d = 2, \\ 0.3 & d = 3, \\ 0.1 & d = 4, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$C = \begin{cases} 90 & \text{for 1-day service,} \\ 70 & \text{for 2-day service,} \\ 40 & \text{for 3-day service,} \\ 40 & \text{for 4-day service.} \end{cases}$$

(a) What is the expected waiting time $\mu_D = E[D]$?

(b) What is the expected deviation $E[D - \mu_D]$?

(c) Express C as a function of D.

(d) What is the expected value $E[C]$?

Problem 2.7.4 Solution

Given the distributions of D, the waiting time in days and the resulting cost, C, we can answer the following questions.

(a) The expected waiting time is simply the expected value of D.

$$E[D] = \sum_{d=1}^{4} d \cdot P_D(d) = 1(0.2) + 2(0.4) + 3(0.3) + 4(0.1) = 2.3 \tag{1}$$

(b) The expected deviation from the waiting time is

$$E[D - \mu_D] = E[D] - E[\mu_d] = \mu_D - \mu_D = 0 \tag{2}$$

(c) C can be expressed as a function of D in the following manner.

$$C(D) = \begin{cases} 90 & D = 1 \\ 70 & D = 2 \\ 40 & D = 3 \\ 40 & D = 4 \end{cases} \tag{3}$$

(d) The expected service charge is

$$E[C] = 90(0.2) + 70(0.4) + 40(0.3) + 40(0.1) = 62 \text{ dollars} \tag{4}$$

Problem 2.8.1 •

In an experiment to monitor two calls, the PMF of N, the number of voice calls, is

$$P_N(n) = \begin{cases} 0.2 & n = 0, \\ 0.7 & n = 1, \\ 0.1 & n = 2, \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find $E[N]$, the expected number of voice calls.

(b) Find $E[N^2]$, the second moment of N.

(c) Find $\text{Var}[N]$, the variance of N.

(d) Find σ_N, the standard deviation of N.

Problem 2.8.1 Solution
Given the following PMF

$$P_N(n) = \begin{cases} 0.2 & n = 0 \\ 0.7 & n = 1 \\ 0.1 & n = 2 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

(a) $E[N] = (0.2)0 + (0.7)1 + (0.1)2 = 0.9$

(b) $E[N^2] = (0.2)0^2 + (0.7)1^2 + (0.1)2^2 = 1.1$

(c) $\text{Var}[N] = E[N^2] - E[N]^2 = 1.1 - (0.9)^2 = 0.29$

(d) $\sigma_N = \sqrt{\text{Var}[N]} = \sqrt{0.29}$

Problem 2.8.12 •
For the delay D in Problem 2.7.4, what is the standard deviation σ_D of the waiting time?

Problem 2.8.12 Solution
The standard deviation can be expressed as

$$\sigma_D = \sqrt{\text{Var}[D]} = \sqrt{E[D^2] - E[D]^2} \tag{1}$$

where

$$E\left[D^2\right] = \sum_{d=1}^{4} d^2 P_D(d) = 0.2 + 1.6 + 2.7 + 1.6 = 6.1 \tag{2}$$

So finally we have

$$\sigma_D = \sqrt{6.1 - 2.3^2} = \sqrt{0.81} = 0.9 \tag{3}$$

Problem 2.9.1 •
In Problem 2.4.1, find $P_{Y|B}(y)$, where the condition $B = \{Y < 3\}$. What are $E[Y|B]$ and $\text{Var}[Y|B]$?

Problem 2.9.1 Solution
From the solution to Problem 2.4.1, the PMF of Y is

$$P_Y(y) = \begin{cases} 1/4 & y = 1 \\ 1/4 & y = 2 \\ 1/2 & y = 3 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The probability of the event $B = \{Y < 3\}$ is $P[B] = 1 - P[Y = 3] = 1/2$. From Theorem 2.17, the conditional PMF of Y given B is

$$P_{Y|B}(y) = \begin{cases} \frac{P_Y(y)}{P[B]} & y \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1/2 & y = 1 \\ 1/2 & y = 2 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The conditional first and second moments of Y are

$$E[Y|B] = \sum_y y P_{Y|B}(y) = 1(1/2) + 2(1/2) = 3/2 \tag{3}$$

$$E[Y^2|B] = \sum_y y^2 P_{Y|B}(y) = 1^2(1/2) + 2^2(1/2) = 5/2 \tag{4}$$

The conditional variance of Y is

$$\text{Var}[Y|B] = E[Y^2|B] - (E[Y|B])^2 = 5/2 - 9/4 = 1/4 \tag{5}$$

Problem 2.9.3 •

In Problem 2.4.3, find $P_{X|B}(x)$, where the condition $B = \{X > 0\}$. What are $E[X|B]$ and $\text{Var}[X|B]$?

Problem 2.9.3 Solution

From the solution to Problem 2.4.3, the PMF of X is

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

The event $B = \{X > 0\}$ has probability $P[B] = P_X(5) + P_X(7) = 0.6$. From Theorem 2.17, the conditional PMF of X given B is

$$P_{X|B}(x) = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2/3 & x = 5 \\ 1/3 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

The conditional first and second moments of X are

$$E[X|B] = \sum_x x P_{X|B}(x) = 5(2/3) + 7(1/3) = 17/3 \tag{3}$$

$$E[X^2|B] = \sum_x x^2 P_{X|B}(x) = 5^2(2/3) + 7^2(1/3) = 33 \tag{4}$$

The conditional variance of X is

$$\text{Var}[X|B] = E[X^2|B] - (E[X|B])^2 = 33 - (17/3)^2 = 8/9 \tag{5}$$

Problem 2.10.1 •

Let X be a binomial (n, p) random variable with $n = 100$ and $p = 0.5$. Let E_2 denote the event that X is a perfect square. Calculate $P[E_2]$.

Problem 2.10.1 Solution

For a binomial (n, p) random variable X, the probability of the event that X is a perfect square is

$$P[E_2] = \sum_{k=0}^{\lfloor\sqrt{n}\rfloor} P_X\left(k^2\right), \tag{1}$$

where

$$P_X(x) = \binom{n}{x} p^x (1-p)^{n-x}, \tag{2}$$

n is a positive integer, and $p \in (0, 1)$.

Here's a matlab function that can be used to calculate this value.

```
%
% function f2_10_1 determines the probability that a binomial
% random variable having parameters n and p takes a value that
% is a perfect square.                                   2/01/06 sk
%

function pe2 = f2_10_1(n,p);
% first determine values of P_x(x) for the squares
for index = 0:10,
   pxx(index+1) = choose(100,index^2)*p^(index^2)*(1-p)^(n-index^2);
end;
% then sum them (events are disjoint)
pe2 = sum(pxx);

%------------------------------------------------------------------

function [num_com] =  choose(n,x);
num_com = factorial(n)/(factorial(x)*factorial(n-x));
```

The output is

```
>> f2_10_1(100,1/2)

ans =

    0.0811
```

Problem 2.10.2 •

Write a Matlab function `x=faxlength8(m)` that produces m random sample values of the fax length X with PMF given in Example 2.29.

Problem 2.10.2 Solution

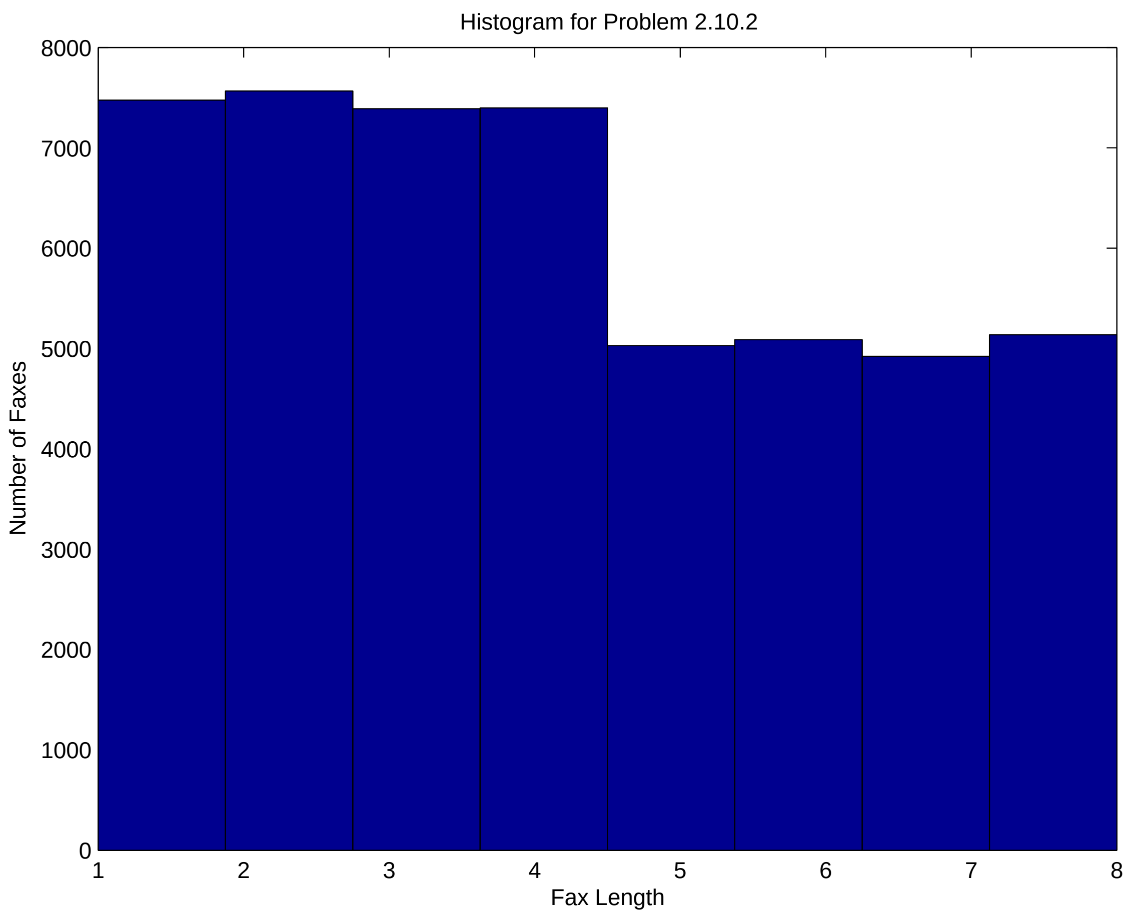
The random variable X given in Example 2.29 is a finite random variable. We can generate random samples using the following code.

```
%
% generates m random samples of the "number of fax pages" random
% variable defined in Example 2.29 on p. 73 of Y&G. 1/31/06
%

function [x] = my_faxlength8(m);
t = rand(m,1);
for index = 1:m,
    if t(index) < 0.15
         x(index) = 1;
    elseif t(index) < 0.3
         x(index) = 2;
    elseif t(index) < 0.45
         x(index) = 3;
    elseif t(index) < 0.6
         x(index) = 4;
    elseif t(index) < 0.7
         x(index) = 5;
    elseif t(index) < 0.8
         x(index) = 6;
    elseif t(index) < 0.9
         x(index) = 7;
    else
         x(index) = 8;
    end;
end;
hist(x,8)
title('Histogram for Problem 2.10.2')
xlabel('Fax Length')
ylabel('Number of Faxes')
disp(['Mean:  ',num2str(mean(x))])
disp(['Variance:  ',num2str(var(x))])
```

Here's the output, including a calculation of the mean and variance of the given PMF. We see that our mean and variance of the experimental data is reasonably close to the predicted values.

```
>> x=my_faxlength8(50);
Mean:  4.06
Variance:  4.9555

>> truemean = .15*(1+2+3+4)+.1*(5+6+7+8)

truemean =

    4.1000

>> truevar = .15*(1^2+2^2+3^2+4^2)+.1*(5^2+6^2+7^2+8^2) - truemean^2

truevar =

    5.0900
```

Here's a histogram for 50,000 samples. (Truth in advertising: It didn't match the distribution so nicely when I tried 50, 500, and 5000 samples!)



Problem 2.10.3 •

For $m = 10$, $m = 100$, and $m = 1000$, use MATLAB to find the average cost of sending m faxes using the model of Example 2.29. Your program input should have the number of trials m as the input. The output should be $\overline{Y} = \frac{1}{m}\sum_{i=1}^{m} Y_i$ where Y_i is the cost of the ith fax. As m becomes large, what do you observe?

Problem 2.10.3 Solution

First we use `faxlength8` from Problem 2.10.2 to generate m samples of the faqx length X. Next we convert that to m samples of the fax cost Y. Summing these samples and dividing by m, we obtain the average cost of m samples. Here is the code:

```
%
% calculate fax costs for problem YG 2.10.3
% 2/01/06 sk
%

for index = 1:max(size(x)),
    if x(index) < 6,
        cost(index) = 10.5*x(index)-0.5*x(index)^2; %%% using (2.65) page 71
    else
        cost(index) = 50;
    end;
end;
mean_cost = sum(cost)/max(size(cost))
```

```
>> clear x; x = my_faxlength8(10); calcost
Mean:  4.3
Variance:  4.6778

mean_cost =

   32.5997

>> clear x; x = my_faxlength8(100); calcost
Mean:  4.09
Variance:  5.2948

mean_cost =

   32.5963

>> clear x; x = my_faxlength8(1000); calcost
Mean:  4.177
Variance:  5.0727

mean_cost =

   32.6132

>> clear x; x = my_faxlength8(10000); calcost
Mean:  4.0698
Variance:  5.063

mean_cost =

   32.5195
```

As the number of samples increases, we should see the mean approach the true mean of the distribution. However, we must remember that these are random samples. While it is unlikely that we will generate an "unusual" sample, it is not impossible. Hence, we should not be too alarmed that the approach to the true mean was not monotone in the number of samples in our test. (By monotone I mean strictly increasing numbers of samples corresponding to strictly decreasing differences between the sample mean and the true mean.)

In a later chapter, we will develop techniques to show how $\overline{Y}$ converges to $E[Y]$ as $m \to \infty$.