Solutions to HW12

Note: These solutions are D. J. Goodman, the authors of our textbook. I have annotated and corrected them as necessary. Text in italics is mine.

Problem 10.10.2 •

Let \( A \) be a nonnegative random variable that is independent of any collection of samples \( X(t_1), \ldots, X(t_k) \) of a wide sense stationary random process \( X(t) \). Is \( Y(t) = A + X(t) \) a wide sense stationary process?

**Problem 10.10.2 Solution**

To show that \( Y(t) \) is wide-sense stationary we must show that it meets the two requirements of Definition 10.15, namely that its expected value and autocorrelation function must be independent of \( t \). Since \( Y(t) = A + X(t) \), the mean of \( Y(t) \) is

\[
E[Y(t)] = E[A] + E[X(t)] = E[A] + \mu_X
\]  

(1)

The autocorrelation of \( Y(t) \) is

\[
R_Y(t, \tau) = E[(A + X(t))(A + X(t + \tau))] 
\]  

(2)

\[
= E[A^2] + E[A]E[X(t)] + AE[X(t + \tau)] + E[X(t)X(t + \tau)] 
\]  

(3)

\[
= E[A^2] + 2E[A]\mu_X + R_X(\tau), 
\]  

(4)

where the last equality is justified by the fact that we are given that \( X(t) \) is wide sense stationary. We see that neither \( E[Y(t)] \) nor \( R_Y(t, \tau) \) depend on \( t \). Thus \( Y(t) \) is a wide sense stationary process.

Problem 10.11.1 •

\( X(t) \) and \( Y(t) \) are independent wide sense stationary processes with expected values \( \mu_X \) and \( \mu_Y \) and autocorrelation functions \( R_X(\tau) \) and \( R_Y(\tau) \) respectively. Let \( W(t) = X(t)Y(t) \).

(a) Find \( \mu_W \) and \( R_W(t, \tau) \) and show that \( W(t) \) is wide sense stationary.

(b) Are \( W(t) \) and \( X(t) \) jointly wide sense stationary?

**Problem 10.11.1 Solution**

\( a) \) Since \( X(t) \) and \( Y(t) \) are independent processes,

\[
E[W(t)] = E[X(t)Y(t)] = E[X(t)]E[Y(t)] = \mu_X\mu_Y. 
\]  

(1)

In addition,

\[
R_W(t, \tau) = E[W(t)W(t + \tau)] 
\]  

(2)

\[
= E[X(t)Y(t)X(t + \tau)Y(t + \tau)] 
\]  

(3)

\[
= E[X(t)X(t + \tau)]E[Y(t)Y(t + \tau)] 
\]  

(4)

\[
= R_X(\tau)R_Y(\tau) 
\]  

(5)

We can conclude that \( W(t) \) is wide sense stationary.
(b) To examine whether \( X(t) \) and \( W(t) \) are jointly wide sense stationary, we calculate
\[
R_{WX}(t, \tau) = E[W(t)X(t+\tau)] = E[X(t)Y(t)X(t+\tau)].
\] (6)

By independence of \( X(t) \) and \( Y(t) \),
\[
R_{WX}(t, \tau) = E[X(t)X(t+\tau)] E[Y(t)] = \mu_Y R_X(\tau).
\] (7)

Since \( W(t) \) and \( X(t) \) are both wide sense stationary and since \( R_{WX}(t, \tau) \) depends only on the time difference \( \tau \), we can conclude from Definition 10.18 that \( W(t) \) and \( X(t) \) are jointly wide sense stationary.

**Problem 10.11.2**

\( X(t) \) is a wide sense stationary random process. For each process \( X_i(t) \) defined below, determine whether \( X_i(t) \) and \( X(t) \) are jointly wide sense stationary.

(a) \( X_1(t) = X(t+a) \)

(b) \( X_2(t) = X(at) \)

**Problem 10.11.2 Solution**

To show that \( X(t) \) and \( X_i(t) \) are jointly wide sense stationary, we must first show that \( X_i(t) \) is wide sense stationary and then we must show that the cross correlation \( R_{XX_i}(t, \tau) \) is only a function of the time difference \( \tau \). For each \( X_i(t) \), we have to check whether these facts are implied by the fact that \( X(t) \) is wide sense stationary.

(a) Since \( E[X_1(t)] = E[X(t+a)] = \mu_X \) and
\[
R_{X_1}(t, \tau) = E[X_1(t)X_1(t+\tau)]
= E[X(t+a)X(t+\tau+a)] 
= R_X(\tau),
\] (1)

we have verified that \( X_1(t) \) is wide sense stationary. Now we calculate the cross correlation
\[
R_{XX_1}(t, \tau) = E[X(t)X_1(t+\tau)]
= E[X(t)X(t+\tau+a)]
= R_X(\tau + a).
\] (6)

Since \( R_{XX_1}(t, \tau) \) depends on the time difference \( \tau \) but not on the absolute time \( t \), we conclude that \( X(t) \) and \( X_1(t) \) are jointly wide sense stationary.

(b) Since \( E[X_2(t)] = E[X(at)] = \mu_X \) and
\[
R_{X_2}(t, \tau) = E[X_2(t)X_2(t+\tau)]
= E[X(at)X(a(t+\tau))]
= E[X(at)X(at+a\tau)] = R_X(a\tau),
\] (9)
we have verified that $X_2(t)$ is wide sense stationary. Now we calculate the cross correlation

$$R_{XX_2}(t, \tau) = E[X(t)X_2(t + \tau)]$$

$$= E[X(t)X(a(t + \tau))]$$

$$= R_X((a - 1)t + \tau).$$

Except for the trivial case when $a = 1$ and $X_2(t) = X(t)$, $R_{XX_2}(t, \tau)$ depends on both the absolute time $t$ and the time difference $\tau$, we conclude that $X(t)$ and $X_2(t)$ are not jointly wide sense stationary.

**Problem 10.11.3**

$X(t)$ is a wide sense stationary stochastic process with autocorrelation function $R_X(\tau) = 10 \sin(2\pi 1000\tau)/(2\pi 1000\tau)$. The process $Y(t)$ is a version of $X(t)$ delayed by 50 microseconds: $Y(t) = X(t - t_0)$ where $t_0 = 5 \times 10^{-5}s$.

(a) Derive the autocorrelation function of $Y(t)$.

(b) Derive the cross-correlation function of $X(t)$ and $Y(t)$.

(c) Is $Y(t)$ wide sense stationary?

(d) Are $X(t)$ and $Y(t)$ jointly wide sense stationary?

**Problem 10.11.3 Solution**

(a) $Y(t)$ has autocorrelation function

$$R_Y(t, \tau) = E[Y(t)Y(t + \tau)]$$

$$= E[X(t - t_0)X(t + \tau - t_0)]$$

$$= R_X(\tau).$$

(b) The cross correlation of $X(t)$ and $Y(t)$ is

$$R_{XY}(t, \tau) = E[X(t)Y(t + \tau)]$$

$$= E[X(t)X(t + \tau - t_0)]$$

$$= R_X(\tau - t_0).$$

(c) We have already verified that $R_Y(t, \tau)$ depends only on the time difference $\tau$. Since $E[Y(t)] = E[X(t - t_0)] = \mu_X$, we have verified that $Y(t)$ is wide sense stationary.

(d) Since $X(t)$ and $Y(t)$ are wide sense stationary and since we have shown that $R_{XY}(t, \tau)$ depends only on $\tau$, we know that $X(t)$ and $Y(t)$ are jointly wide sense stationary.

**Comment:** This problem is badly designed since the conclusions don’t depend on the specific $R_X(\tau)$ given in the problem text. (Sorry about that!)
Problem 11.2.1

The random sequence $X_n$ is the input to a discrete-time filter. The output is

$$Y_n = \frac{X_{n+1} + X_n + X_{n-1}}{3}.$$  

(a) What is the impulse response $h_n$?

(b) Find the autocorrelation of the output $Y_n$ when $X_n$ is a wide sense stationary random sequence with $\mu_X = 0$ and autocorrelation

$$R_X[n] = \begin{cases} 
1 & n = 0, \\
0 & \text{otherwise}. 
\end{cases}$$

Problem 11.2.1 Solution

(a) Note that

$$Y_i = \sum_{n=-\infty}^{\infty} h_n X_{i-n} = \frac{1}{3} X_{i+1} + \frac{1}{3} X_i + \frac{1}{3} X_{i-1} \quad (1)$$

By matching coefficients, we see that

$$h_n = \begin{cases} 
1/3 & n = -1, 0, 1 \\
0 & \text{otherwise} 
\end{cases} \quad (2)$$

(b) By Theorem 11.5, the output autocorrelation is

$$R_Y[n] = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} h_i h_j R_X[n+i-j] \quad (3)$$

$$= \frac{1}{9} \sum_{i=-1}^{1} \sum_{j=-1}^{1} R_X[n+i-j] \quad (4)$$

$$= \frac{1}{9} \left( R_X[n+2] + 2R_X[n+1] + 3R_X[n] + 2R_X[n-1] + R_X[n-2] \right) \quad (5)$$

We see that the filter is linear and time invariant. Substituting in $R_X[n]$ yields

$$R_Y[n] = \begin{cases} 
1/3 & n = 0 \\
2/9 & |n| = 1 \\
1/9 & |n| = 2 \\
0 & \text{otherwise} 
\end{cases} \quad (6)$$
Problem 11.2.2 •

$X(t)$ is a wide sense stationary process with autocorrelation function

$$R_X(\tau) = 10 \frac{\sin(2000\pi t) + \sin(1000\pi t)}{2000\pi t}.$$ 

The process $X(t)$ is sampled at rate $1/T_s = 4,000$ Hz, yielding the discrete-time process $X_n$. What is the autocorrelation function $R_X[k]$ of $X_n$?

**Problem 11.2.2 Solution**

Applying Theorem 11.4 with sampling period $T_s = 1/4000$ s yields

$$R_X[k] = R_X(kT_s) = 10 \frac{\sin(2000\pi kT_s) + \sin(1000\pi kT_s)}{\pi k}$$

$$= 20 \sin(0.5\pi k) + \sin(0.25\pi k)$$

$$= 10 \text{sinc}(0.5k) + 5 \text{sinc}(0.25k)$$

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Problem 11.3.1 •

$X_n$ is a stationary Gaussian sequence with expected value $E[X_n] = 0$ and autocorrelation function $R_X[k] = 2^{-|k|}$. Find the PDF of $X = [X_1 \ X_2 \ X_3]'$.

**Problem 11.3.1 Solution**

Since the process $X_n$ has expected value $E[X_n] = 0$, we know that $C_X(k) = R_X(k) = 2^{-|k|}$. Thus $X = [X_1 \ X_2 \ X_3]'$ has covariance matrix

$$C_X = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \\ 1/2 & 1/2 & 1 \end{bmatrix}.$$ (1)

From Definition 5.17, the PDF of $X$ is

$$f_X(x) = \frac{1}{(2\pi)^{n/2} |\det(C_X)|^{1/2}} \exp \left(-\frac{1}{2} x' C_X^{-1} x\right).$$ (2)

Equivalently, we can write out the PDF in terms of the variables $x_1$, $x_2$ and $x_3$. To do so we find that the inverse covariance matrix is

$$C_X^{-1} = \begin{bmatrix} 4/3 & -2/3 & 0 \\ -2/3 & 5/3 & -2/3 \\ 0 & -2/3 & 4/3 \end{bmatrix}.$$ (3)

A little bit of algebra will show that $\det(C_X) = 9/16$ and that

$$\frac{1}{2} x' C_X^{-1} x = \frac{2x_1^2}{3} + \frac{5x_2^2}{6} + \frac{2x_3^2}{3} - \frac{2x_1x_2}{3} - \frac{2x_2x_3}{3}.$$ (4)

It follows that

$$f_X(x) = \frac{4}{3(2\pi)^{3/2}} \exp \left(-\frac{2x_1^2}{3} - \frac{5x_2^2}{6} - \frac{2x_3^2}{3} + \frac{2x_1x_2}{3} + \frac{2x_2x_3}{3}\right).$$ (5)
Problem 11.3.2

\[ X_n \] is a sequence of independent random variables such that \( X_n = 0 \) for \( n < 0 \) while for \( n \geq 0 \), each \( X_n \) is a Gaussian \((0, 1)\) random variable. Passing \( X_n \) through the filter \( h = [1 \ -1 \ 1]^\prime \) yields the output \( Y_n \). Find the PDFs of:

(a) \( Y_3 = [Y_1 \ Y_2 \ Y_3]^\prime \),

(b) \( Y_2 = [Y_1 \ Y_2]^\prime \).

Problem 11.3.2 Solution

The sequence \( X_n \) is passed through the filter

\[ h = [h_0 \ h_1 \ h_2]^\prime = [1 \ -1 \ 1]^\prime \]  

The output sequence is \( Y_n \).

(a) Following the approach of Equation (11.58), we can write the output \( Y_3 = [Y_1 \ Y_2 \ Y_3]^\prime \) as

\[
Y_3 = \begin{bmatrix} Y_1 \\ Y_2 \\ Y_3 \end{bmatrix} = \begin{bmatrix} h_1 & h_0 & 0 \\ h_2 & h_1 & h_0 \\ 0 & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix} = HX. \]

We note that the components of \( X \) are iid Gaussian \((0, 1)\) random variables. Hence \( X \) has covariance matrix \( C_X = I \), the identity matrix. Since \( Y_3 = HX \),

\[ C_Y = HC_X H' = HH' = \begin{bmatrix} 2 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 3 \end{bmatrix}. \]

Some calculation (by hand or by MATLAB) will show that \( \det(C_Y) = 3 \) and that

\[ C_Y^{-1} = \frac{1}{3} \begin{bmatrix} 5 & 4 & 1 \\ 4 & 5 & 2 \\ 1 & 2 & 2 \end{bmatrix}. \]

Some algebra will show that

\[ y' C_Y^{-1} y = \frac{5y_1^2 + 5y_2^2 + 2y_3^2 + 8y_1y_2 + 2y_1y_3 + 4y_2y_3}{3}. \]

This implies \( Y_3 \) has PDF

\[ f_{Y_3}(y) = \frac{1}{(2\pi)^{3/2} [\det(C_Y)]^{1/2}} \exp \left( -\frac{1}{2} y' C_Y^{-1} y \right) \]

\[ = \frac{1}{(2\pi)^{3/2} \sqrt{3}} \exp \left( -\frac{5y_1^2 + 5y_2^2 + 2y_3^2 + 8y_1y_2 + 2y_1y_3 + 4y_2y_3}{6} \right). \]
(b) To find the PDF of \( Y_2 = [Y_1 \ Y_2]' \), we start by observing that the covariance matrix of \( Y_2 \) is just the upper left 2 \times 2 submatrix of \( C_{Y_2} \). That is,

\[
C_{Y_2} = \begin{bmatrix} 2 & -2 \\ -2 & 3 \end{bmatrix} \quad \text{and} \quad C_{Y_2}^{-1} = \begin{bmatrix} 3/2 & 1 \\ 1 & 1 \end{bmatrix}.
\]  

Since \( \det(C_{Y_2}) = 2 \), it follows that

\[
f_{Y_2}(y) = \frac{1}{(2\pi)^{3/2}\sqrt{\det(C_{Y_2})}} \exp\left(-\frac{1}{2} y' C_{Y_2}^{-1} y\right)
\]

\[
= \frac{1}{(2\pi)^{3/2}\sqrt{2}} \exp\left(-\frac{3}{2} y_1^2 - 2y_1y_2 - y_2^2\right).
\]

Problem 11.5.1 •

(1) \( X(t) \) is a wide sense stationary process with autocorrelation function

\[
R_X(\tau) = 10\sin(2000\pi \tau) + \sin(1000\pi \tau) / 2000\pi \tau.
\]

What is the power spectral density of \( X(t) \)?

**Problem 11.5.1 Solution**

To use Table 11.1, we write \( R_X(\tau) \) in terms of the autocorrelation

\[
sinc(x) = \frac{\sin(\pi x)}{\pi x}.
\]

In terms of the sinc(\( \cdot \)) function, we obtain

\[
R_X(\tau) = 10 \text{sinc}(2000\tau) + 5 \text{sinc}(1000\tau).
\]

From Table 11.1,

\[
S_X(f) = \frac{10}{2000} \text{rect} \left( \frac{f}{2000} \right) + \frac{5}{1000} \text{rect} \left( \frac{f}{1000} \right)
\]

Here is a graph of the PSD.
Problem 11.6.1

$X_n$ is a wide sense stationary discrete-time random sequence with autocorrelation function

$$R_X[k] = \begin{cases} 
\delta[k] + (0.1)^{|k|} & k = 0, \pm 1, \pm 2, \ldots, \\
0 & \text{otherwise.}
\end{cases}$$

Find the power spectral density $S_X(f)$.

Problem 11.6.1 Solution

Since the random sequence $X_n$ has autocorrelation function

$$R_X[k] = \delta[k] + (0.1)^{|k|}, \quad (1)$$

We can find the PSD directly from Table 11.2 with $0.1^{|k|}$ corresponding to $a^{|k|}$. The table yields

$$S_X(\phi) = 1 + \frac{1 - (0.1)^2}{1 + (0.1)^2 - 2(0.1)\cos 2\pi\phi} = \frac{2 - 0.2 \cos 2\pi\phi}{1.01 - 0.2 \cos 2\pi\phi}. \quad (2)$$