

**Problem 7.4.1 •**

$X_1, \dots, X_n$  are  $n$  independent identically distributed samples of random variable  $X$  with PMF

$$P_X(x) = \begin{cases} 0.1 & x = 0, \\ 0.9 & x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

- (a) How is  $E[X]$  related to  $P_X(1)$ ?
- (b) Use Chebyshev's inequality to find the confidence level  $\alpha$  such that  $M_{90}(X)$ , the estimate based on 90 observations, is within 0.05 of  $P_X(1)$ . In other words, find  $\alpha$  such that

$$P[|M_{90}(X) - P_X(1)| \geq 0.05] \leq \alpha.$$

- (c) Use Chebyshev's inequality to find out how many samples  $n$  are necessary to have  $M_n(X)$  within 0.03 of  $P_X(1)$  with confidence level 0.1. In other words, find  $n$  such that

$$P[|M_n(X) - P_X(1)| \geq 0.03] \leq 0.1.$$

**Problem 7.4.1 Solution**

We are given that  $X_1, \dots, X_n$  are  $n$  independent identically distributed samples of the random variable  $X$  having PMF

$$P_X(x) = \begin{cases} 0.1 & x = 0 \\ 0.9 & x = 1 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (a)  $E[X]$  is in fact the same as  $P_X(1)$  because  $X$  is a Bernoulli random variable.
- (b) By Chebyshev's inequality,

$$P[|M_{90}(X) - P_X(1)| \geq .05] = P[|M_{90}(X) - E[X]| \geq .05] \leq \frac{\text{Var}[X]}{(0.5)^2} = \alpha \quad (2)$$

so

$$\alpha = \frac{\sigma_X^2}{90(.05)^2} = \frac{.09}{90(.05)^2} = 0.4 \quad (3)$$

- (c) Now we wish to find the value of  $n$  such that  $P[|M_n(X) - P_X(1)| \geq .03] \leq .1$ . From the Chebyshev inequality, we write

$$0.1 = \frac{\sigma_X^2}{n(.03)^2}. \quad (4)$$

Since  $\sigma_X^2 = 0.09$ , solving for  $n$  yields  $n = 1000$ .

**Problem 7.4.2 •**

Let  $X_1, X_2, \dots$  denote an iid sequence of random variables, each with expected value 75 and standard deviation 15.

- How many samples  $n$  do we need to guarantee that the sample mean  $M_n(X)$  is between 74 and 76 with probability 0.99?
- If each  $X_i$  has a Gaussian distribution, how many samples  $n'$  would we need to guarantee  $M_{n'}(X)$  is between 74 and 76 with probability 0.99?

**Problem 7.4.2 Solution**

$X_1, X_2, \dots$  are iid random variables each with mean 75 and standard deviation 15.

- We would like to find the value of  $n$  such that

$$P[74 \leq M_n(X) \leq 76] = 0.99 \quad (1)$$

When we know only the mean and variance of  $X_i$ , our only real tool is the Chebyshev inequality which says that

$$P[74 \leq M_n(X) \leq 76] = 1 - P[|M_n(X) - E[X]| \geq 1] \quad (2)$$

$$\geq 1 - \frac{\text{Var}[X]}{n} = 1 - \frac{225}{n} \geq 0.99 \quad (3)$$

This yields  $n \geq 22,500$ .

- If each  $X_i$  is a Gaussian, the sample mean,  $M_n(X)$  will also be Gaussian with mean and variance

$$E[M_{n'}(X)] = E[X] = 75 \quad (4)$$

$$\text{Var}[M_{n'}(X)] = \text{Var}[X]/n' = 225/n' \quad (5)$$

In this case,

$$P[74 \leq M_{n'}(X) \leq 76] = \Phi\left(\frac{76 - \mu}{\sigma}\right) - \Phi\left(\frac{74 - \mu}{\sigma}\right) \quad (6)$$

$$= \Phi(\sqrt{n'}/15) - \Phi(-\sqrt{n'}/15) \quad (7)$$

$$= 2\Phi(\sqrt{n'}/15) - 1 = 0.99 \quad (8)$$

so  $\Phi(\sqrt{n'}/15) = 1.99/2 = .995$ . Then from the table,  $\sqrt{n'}/15 \approx 2.58$  so  $n' \approx 1,498$ .

Since even under the Gaussian assumption, the number of samples  $n'$  is so large that even if the  $X_i$  are not Gaussian, the sample mean may be approximated by a Gaussian. Hence, about 1500 samples probably is about right. However, in the absence of any information about the PDF of  $X_i$  beyond the mean and variance, we cannot make any guarantees stronger than that given by the Chebyshev inequality.

**Problem 7.4.3 •**

Let  $X_A$  be the indicator random variable for event  $A$  with probability  $P[A] = 0.8$ . Let  $\hat{P}_n(A)$  denote the relative frequency of event  $A$  in  $n$  independent trials.

- (a) Find  $E[X_A]$  and  $\text{Var}[X_A]$ .
- (b) What is  $\text{Var}[\hat{P}_n(A)]$ ?
- (c) Use the Chebyshev inequality to find the confidence coefficient  $1 - \alpha$  such that  $\hat{P}_{100}(A)$  is within 0.1 of  $P[A]$ . In other words, find  $\alpha$  such that

$$P \left[ \left| \hat{P}_{100}(A) - P[A] \right| \leq 0.1 \right] \geq 1 - \alpha.$$

- (d) Use the Chebyshev inequality to find out how many samples  $n$  are necessary to have  $\hat{P}_n(A)$  within 0.1 of  $P[A]$  with confidence coefficient 0.95. In other words, find  $n$  such that

$$P \left[ \left| \hat{P}_n(A) - P[A] \right| \leq 0.1 \right] \geq 0.95.$$

**Problem 7.4.3 Solution**

- (a) Since  $X_A$  is a Bernoulli ( $p = P[A]$ ) random variable,

$$E[X_A] = P[A] = 0.8, \quad \text{Var}[X_A] = P[A](1 - P[A]) = 0.16. \quad (1)$$

- (b) Let  $X_{A,i}$  denote  $X_A$  for the  $i$ th trial. Since  $\hat{P}_n(A) = M_n(X_A) = \frac{1}{n} \sum_{i=1}^n X_{A,i}$ ,

$$\text{Var}[\hat{P}_n(A)] = \frac{1}{n^2} \sum_{i=1}^n \text{Var}[X_{A,i}] = \frac{P[A](1 - P[A])}{n}. \quad (2)$$

- (c) Since  $\hat{P}_{100}(A) = M_{100}(X_A)$ , we can use Theorem 7.12(b) to write

$$P \left[ \left| \hat{P}_{100}(A) - P[A] \right| < c \right] \geq 1 - \frac{\text{Var}[X_A]}{100c^2} = 1 - \frac{0.16}{100c^2} = 1 - \alpha. \quad (3)$$

For  $c = 0.1$ ,  $\alpha = 0.16/[100(0.1)^2] = 0.16$ . Thus, with 100 samples, our confidence coefficient is  $1 - \alpha = 0.84$ .

- (d) In this case, the number of samples  $n$  is unknown. Once again, we use Theorem 7.12(b) to write

$$P \left[ \left| \hat{P}_n(A) - P[A] \right| < c \right] \geq 1 - \frac{\text{Var}[X_A]}{nc^2} = 1 - \frac{0.16}{nc^2} = 1 - \alpha. \quad (4)$$

For  $c = 0.1$ , we have confidence coefficient  $1 - \alpha = 0.95$  if  $\alpha = 0.16/[n(0.1)^2] = 0.05$ , or  $n = 320$ .

**Problem 7.4.4 •**

$X$  is a Bernoulli random variable with unknown success probability  $p$ . Using 100 independent samples of  $X$  find a confidence interval estimate of  $p$  with confidence coefficient 0.99. If  $M_{100}(X) = 0.06$ , what is our interval estimate?

**Problem 7.4.4 Solution**

Since  $E[X] = \mu_X = p$  and  $\text{Var}[X] = p(1-p)$ , we use Theorem 7.12(b) to write

$$P[|M_{100}(X) - p| < c] \geq 1 - \frac{p(1-p)}{100c^2} = 1 - \alpha. \quad (1)$$

For confidence coefficient 0.99, we require

$$\frac{p(1-p)}{100c^2} \leq 0.01 \quad \text{or} \quad c \geq \sqrt{p(1-p)}. \quad (2)$$

Since  $p$  is unknown, we must ensure that the constraint is met for every value of  $p$ . The worst case occurs at  $p = 1/2$  which maximizes  $p(1-p)$ . In this case,  $c = \sqrt{1/4} = 1/2$  is the smallest value of  $c$  for which we have confidence coefficient of at least 0.99.

If  $M_{100}(X) = 0.06$ , our interval estimate for  $p$  is

$$M_{100}(X) - c < p < M_{100}(X) + c. \quad (3)$$

Since  $p \geq 0$ ,  $M_{100}(X) = 0.06$  and  $c = 0.5$  imply that our interval estimate is

$$0 \leq p < 0.56. \quad (4)$$

Our interval estimate is not very tight because because 100 samples is not very large for a confidence coefficient of 0.99.

**Problem 7.4.5 •**

In  $n$  independent experimental trials, the relative frequency of event  $A$  is  $\hat{P}_n(A)$ . How large should  $n$  be to ensure that the confidence interval estimate

$$\hat{P}_n(A) - 0.05 \leq P[A] \leq \hat{P}_n(A) + 0.05$$

has confidence coefficient 0.9?

**Problem 7.4.5 Solution**

First we observe that the interval estimate can be expressed as

$$\left| \hat{P}_n(A) - P[A] \right| < 0.05. \quad (1)$$

Since  $\hat{P}_n(A) = M_n(X_A)$  and  $E[M_n(X_A)] = P[A]$ , we can use Theorem 7.12(b) to write

$$P \left[ \left| \hat{P}_n(A) - P[A] \right| < 0.05 \right] \geq 1 - \frac{\text{Var}[X_A]}{n(0.05)^2}. \quad (2)$$

Note that  $\text{Var}[X_A] = P[A](1 - P[A]) \leq \max_{x \in (0,1)} x(1 - x) = 0.25$ . Thus for confidence coefficient 0.9, we require that

$$1 - \frac{\text{Var}[X_A]}{n(0.05)^2} \geq 1 - \frac{0.25}{n(0.05)^2} \geq 0.9. \quad (3)$$

This implies  $n \geq 1,000$  samples are needed.

### Problem 7.4.6 ■

When we perform an experiment, event  $A$  occurs with probability  $P[A] = 0.01$ . In this problem, we estimate  $P[A]$  using  $\hat{P}_n(A)$ , the relative frequency of  $A$  over  $n$  independent trials.

- (a) How many trials  $n$  are needed so that the interval estimate

$$\hat{P}_n(A) - 0.001 < P[A] < \hat{P}_n(A) + 0.001$$

has confidence coefficient  $1 - \alpha = 0.99$ ?

- (b) How many trials  $n$  are needed so that the probability  $\hat{P}_n(A)$  differs from  $P[A]$  by more than 0.1% is less than 0.01?

### Problem 7.4.6 Solution

Both questions can be answered using the following equation from Example 7.6:

$$P \left[ \left| \hat{P}_n(A) - P[A] \right| \geq c \right] \leq \frac{P[A](1 - P[A])}{nc^2} \quad (1)$$

The unusual part of this problem is that we are given the true value of  $P[A]$ . Since  $P[A] = 0.01$ , we can write

$$P \left[ \left| \hat{P}_n(A) - P[A] \right| \geq c \right] \leq \frac{0.0099}{nc^2} \quad (2)$$

- (a) In this part, we meet the requirement by choosing  $c = 0.001$  yielding

$$P \left[ \left| \hat{P}_n(A) - P[A] \right| \geq 0.001 \right] \leq \frac{9900}{n} \quad (3)$$

Thus to have confidence level 0.01, we require that  $9900/n \leq 0.01$ . This requires  $n \geq 990,000$ .

- (b) In this case, we meet the requirement by choosing  $c = 10^{-3}P[A] = 10^{-5}$ . This implies

$$P \left[ \left| \hat{P}_n(A) - P[A] \right| \geq c \right] \leq \frac{P[A](1 - P[A])}{nc^2} = \frac{0.0099}{n10^{-10}} = \frac{9.9 \times 10^7}{n} \quad (4)$$

The confidence level 0.01 is met if  $9.9 \times 10^7/n = 0.01$  or  $n = 9.9 \times 10^9$ .

**Problem 7.4.7 ■**

In communication systems, the error probability  $P[E]$  may be difficult to calculate; however it may be easy to derive an upper bound of the form  $P[E] \leq \epsilon$ . In this case, we may still want to estimate  $P[E]$  using the relative frequency  $\hat{P}_n(E)$  of  $E$  in  $n$  trials. In this case, show that

$$P \left[ \left| \hat{P}_n(E) - P[E] \right| \geq c \right] \leq \frac{\epsilon}{nc^2}.$$

**Problem 7.4.7 Solution**

Since the relative frequency of the error event  $E$  is  $\hat{P}_n(E) = M_n(X_E)$  and  $E[M_n(X_E)] = P[E]$ , we can use Theorem 7.12(a) to write

$$P \left[ \left| \hat{P}_n(A) - P[E] \right| \geq c \right] \leq \frac{\text{Var}[X_E]}{nc^2}. \quad (1)$$

Note that  $\text{Var}[X_E] = P[E](1 - P[E])$  since  $X_E$  is a Bernoulli ( $p = P[E]$ ) random variable. Using the additional fact that  $P[E] \leq \epsilon$  and the fairly trivial fact that  $1 - P[E] \leq 1$ , we can conclude that

$$\text{Var}[X_E] = P[E](1 - P[E]) \leq P[E] \leq \epsilon. \quad (2)$$

Thus

$$P \left[ \left| \hat{P}_n(A) - P[E] \right| \geq c \right] \leq \frac{\text{Var}[X_E]}{nc^2} \leq \frac{\epsilon}{nc^2}. \quad (3)$$

**Problem 10.3.1 ■**

Let  $W$  be an exponential random variable with PDF

$$f_W(w) = \begin{cases} e^{-w} & w \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Find the CDF  $F_{X(t)}(x)$  of the time delayed ramp process  $X(t) = t - W$ .

**Problem 10.3.1 Solution**

In this problem, we start from first principles. What makes this problem fairly straightforward is that the ramp is defined for all time. That is, the ramp doesn't start at time  $t = W$ .

$$P[X(t) \leq x] = P[t - W \leq x] = P[W \geq t - x] \quad (1)$$

Since  $W \geq 0$ , if  $x \geq t$  then  $P[W \geq t - x] = 1$ . When  $x < t$ ,

$$P[W \geq t - x] = \int_{t-x}^{\infty} f_W(w) dw = e^{-(t-x)} \quad (2)$$

Combining these facts, we have

$$F_{X(t)}(x) = P[W \geq t - x] = \begin{cases} e^{-(t-x)} & x < t \\ 1 & t \leq x \end{cases} \quad (3)$$

We note that the CDF contain no discontinuities. Taking the derivative of the CDF  $F_{X(t)}(x)$  with respect to  $x$ , we obtain the PDF

$$f_{X(t)}(x) = \begin{cases} e^{x-t} & x < t \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

### Problem 10.3.2 ■

In a production line for 10 kHz oscillators, the output frequency of each oscillator is a random variable  $W$  uniformly distributed between 9980 Hz and 1020 Hz. The frequencies of different oscillators are independent. The oscillator company has an order for one part in  $10^4$  oscillators with frequency between 9999 Hz and 10,001 Hz. A technician takes one oscillator per minute from the production line and measures its exact frequency. (This test takes one minute.) The random variable  $T_r$  minutes is the elapsed time at which the technician finds  $r$  acceptable oscillators.

- What is  $p$ , the probability that any single oscillator has one-part-in- $10^4$  accuracy?
- What is  $E[T_1]$  minutes, the expected time for the technician to find the first one-part-in- $10^4$  oscillator?
- What is the probability that the technician will find the first one-part-in- $10^4$  oscillator in exactly 20 minutes?
- What is  $E[T_5]$ , the expected time of finding the fifth one-part-in- $10^4$  oscillator?

### Problem 10.3.2 Solution

- Each resistor has frequency  $W$  in Hertz with uniform PDF

$$f_R(r) = \begin{cases} 0.025 & 9980 \leq r \leq 1020 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The probability that a test yields a one part in  $10^4$  oscillator is

$$p = P[9999 \leq W \leq 10001] = \int_{9999}^{10001} (0.025) dr = 0.05 \quad (2)$$

- To find the PMF of  $T_1$ , we view each oscillator test as an independent trial. A success occurs on a trial with probability  $p$  if we find a one part in  $10^4$  oscillator. The first one part in  $10^4$  oscillator is found at time  $T_1 = t$  if we observe failures on trials  $1, \dots, t-1$  followed by a success on trial  $t$ . Hence, just as in Example 2.11,  $T_1$  has the geometric PMF

$$P_{T_1}(t) = \begin{cases} (1-p)^{t-1}p & t = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (3)$$

A geometric random variable with success probability  $p$  has mean  $1/p$ . This is derived in Theorem 2.5. The expected time to find the first good oscillator is  $E[T_1] = 1/p = 20$  minutes.

- (c) Since  $p = 0.05$ , the probability the first one part in  $10^4$  oscillator is found in exactly 20 minutes is  $P_{T_1}(20) = (0.95)^{19}(0.05) = 0.0189$ .
- (d) The time  $T_5$  required to find the 5th one part in  $10^4$  oscillator is the number of trials needed for 5 successes.  $T_5$  is a Pascal random variable. If this is not clear, see Example 2.15 where the Pascal PMF is derived. When we are looking for 5 successes, the Pascal PMF is

$$P_{T_5}(t) = \begin{cases} \binom{t-1}{4} p^5 (1-p)^{t-5} & t = 5, 6, \dots \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

Looking up the Pascal PMF in Appendix A, we find that  $E[T_5] = 5/p = 100$  minutes. The following argument is a second derivation of the mean of  $T_5$ . Once we find the first one part in  $10^4$  oscillator, the number of additional trials needed to find the next one part in  $10^4$  oscillator once again has a geometric PMF with mean  $1/p$  since each independent trial is a success with probability  $p$ . Similarly, the time required to find 5 one part in  $10^4$  oscillators is the sum of five independent geometric random variables. That is,

$$T_5 = K_1 + K_2 + K_3 + K_4 + K_5 \quad (5)$$

where each  $K_i$  is identically distributed to  $T_1$ . Since the expectation of the sum equals the sum of the expectations,

$$E[T_5] = E[K_1 + K_2 + K_3 + K_4 + K_5] = 5E[K_i] = 5/p = 100 \text{ minutes} \quad (6)$$

### Problem 10.3.3 ■

For the random process of Problem 10.3.2, what is the conditional PMF of  $T_2$  given  $T_1$ ? If the technician finds the first oscillator in 3 minutes, what is  $E[T_2|T_1 = 3]$ , the conditional expected value of the time of finding the second one-part-in- $10^4$  oscillator?

### Problem 10.3.3 Solution

Once we find the first one part in  $10^4$  oscillator, the number of additional tests needed to find the next one part in  $10^4$  oscillator once again has a geometric PMF with mean  $1/p$  since each independent trial is a success with probability  $p$ . That is  $T_2 = T_1 + T'$  where  $T'$  is independent and identically distributed to  $T_1$ . Thus,

$$E[T_2|T_1 = 3] = E[T_1|T_1 = 3] + E[T'|T_1 = 3] \quad (1)$$

$$= 3 + E[T'] = 23 \text{ minutes.} \quad (2)$$

### Problem 10.4.1 ●

Suppose that at the equator, we can model the noontime temperature in degrees Celsius,  $X_n$ , on day  $n$  by a sequence of iid Gaussian random variables with a mean of 30 degrees and standard deviation of 5 degrees. A new random process  $Y_k = [X_{2k-1} + X_{2k}]/2$  is obtained by averaging the temperature over two days. Is  $Y_k$  an iid random sequence?

**Problem 10.4.1 Solution**

Each  $Y_k$  is the sum of two identical independent Gaussian random variables. Hence, by *Thm. 6.10, p. 253*, each  $Y_k$  is a Gaussian random variable and, since the distribution does not depend on  $k$ , all of the  $Y_k$  must have the same PDF. That is, the  $Y_k$  are identically distributed. Next, we observe that the sequence of  $Y_k$  is independent. To see this, we observe that each  $Y_k$  is composed of two samples of  $X_k$  that are unused by any other  $Y_j$  for  $j \neq k$ . Formally, for  $k \neq l$

$$\text{Cov}[Y_k, Y_l] = E[Y_k Y_l] - E[Y_k] E[Y_l] \quad (1)$$

$$= \frac{1}{4} E[(X_{2k-1} + X_{2k})(X_{2l-1} + X_{2l})] - 30^2 \quad (2)$$

$$= \frac{1}{4} E[(X_{2k-1} X_{2l-1} + X_{2k-1} X_{2l} + X_{2k} X_{2l-1} + X_{2k} X_{2l})] - 900 \quad (3)$$

$$= \frac{1}{4} (E[X_{2k-1}] E[X_{2l-1}] + E[X_{2k-1}] E[X_{2l}] + E[X_{2k}] E[X_{2l-1}] + E[X_{2k}] E[X_{2l}]) - 900 \quad (4)$$

$$= \frac{1}{4} (900 + 900 + 900 + 900) - 900 \quad (5)$$

$$= 0, \quad (6)$$

where the fourth equality follows from the  $X_n$  being independent, so off-diagonal terms of the covariance are zero so the  $Y_k$  are independent.

**Problem 10.4.2 ■**

For the equatorial noontime temperature sequence  $X_n$  of Problem 10.4.1, a second sequence of averaged temperatures is  $W_n = [X_n + X_{n-1}]/2$ . Is  $W_n$  an iid random sequence?

**Problem 10.4.2 Solution**

To be iid, the sequence must be both independent and identically distributed. First, let's look at the distribution. Each  $W_n$  is the sum of two identical independent Gaussian random variables. Hence, each  $W_n$  must have the same PDF. That is, the  $W_n$  are identically distributed. Next we check for independence. Informally, since  $W_{n-1}$  and  $W_n$  both use  $X_{n-1}$  in their averaging,  $W_{n-1}$  and  $W_n$  are dependent. We verify this observation by calculating the covariance of  $W_{n-1}$  and  $W_n$ . First, we observe that for all  $n$ ,

$$E[W_n] = (E[X_n] + E[X_{n-1}])/2 = 30 \quad (1)$$

Next, we observe that  $W_{n-1}$  and  $W_n$  have covariance

$$\text{Cov}[W_{n-1}, W_n] = E[W_{n-1} W_n] - E[W_n] E[W_{n-1}] \quad (2)$$

$$= \frac{1}{4} E[(X_{n-1} + X_{n-2})(X_n + X_{n-1})] - 900 \quad (3)$$

We observe that for  $n \neq m$ ,  $E[X_n X_m] = E[X_n] E[X_m] = 900$  while

$$E[X_n^2] = \text{Var}[X_n] + (E[X_n])^2 = 925 \quad (4)$$

Thus,

$$\text{Cov}[W_{n-1}, W_n] = \frac{900 + 925 + 900 + 900}{4} - 900 = 25/4 \quad (5)$$

Since  $\text{Cov}[W_{n-1}, W_n] \neq 0$ , we conclude that  $W_n$  and  $W_{n-1}$  must be dependent. So  $W_n$  is not an iid sequence.

### Problem 10.4.3 ■

Let  $Y_k$  denote the number of failures between successes  $k-1$  and  $k$  of a Bernoulli ( $p$ ) random process. Also, let  $Y_1$  denote the number of failures before the first success. What is the PMF  $P_{Y_k}(y)$ ? Is  $Y_k$  an iid random sequence?

### Problem 10.4.3 Solution

The number  $Y_k$  of failures between successes  $k-1$  and  $k$  is exactly  $y \geq 0$  iff after success  $k-1$ , there are  $y$  failures followed by a success. Since the Bernoulli trials are independent, the probability of this event is  $(1-p)^y p$ . The complete PMF of  $Y_k$  is

$$P_{Y_k}(y) = \begin{cases} (1-p)^y p & y = 0, 1, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Since this argument is valid for all  $k$  including  $k=1$ , we can conclude that  $Y_1, Y_2, \dots$  are identically distributed. Moreover, since the trials are independent, the failures between successes  $k-1$  and  $k$  and the number of failures between successes  $k'-1$  and  $k'$  are independent. Hence,  $Y_1, Y_2, \dots$  is an iid sequence.

### Problem 10.5.1 ●

The arrivals of new telephone calls at a telephone switching office is a Poisson process  $N(t)$  with an arrival rate of  $\lambda = 4$  calls per second. An experiment consists of monitoring the switching office and recording  $N(t)$  over a 10-second interval.

- What is  $P_{N(1)}(0)$ , the probability of no phone calls in the first second of observation?
- What is  $P_{N(1)}(4)$ , the probability of exactly four calls arriving in the first second of observation?
- What is  $P_{N(2)}(2)$ , the probability of exactly two calls arriving in the first two seconds?

### Problem 10.5.1 Solution

This is a very straightforward problem. The Poisson process has rate  $\lambda = 4$  calls per second. When  $t$  is measured in seconds, each  $N(t)$  is a Poisson random variable with mean  $4t$  and thus has PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(4t)^n}{n!} e^{-4t} & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

Using the general expression for the PMF, we can write down the answer for each part.

- (a)  $P_{N(1)}(0) = 4^0 e^{-4}/0! = e^{-4} \approx 0.0183$ .
- (b)  $P_{N(1)}(4) = 4^4 e^{-4}/4! = 32e^{-4}/3 \approx 0.1954$ .
- (c)  $P_{N(2)}(2) = 8^2 e^{-8}/2! = 32e^{-8} \approx 0.0107$ .

**Problem 10.5.2 •**

Queries presented to a computer database are a Poisson process of rate  $\lambda = 6$  queries per minute. An experiment consists of monitoring the database for  $m$  minutes and recording  $N(m)$ , the number of queries presented. The answer to each of the following questions can be expressed in terms of the PMF  $P_{N(m)}(k) = P[N(m) = k]$ .

- (a) What is the probability of no queries in a one minute interval?
- (b) What is the probability of exactly six queries arriving in a one-minute interval?
- (c) What is the probability of exactly three queries arriving in a one-half-minute interval?

**Problem 10.5.2 Solution**

Following the instructions given, we express each answer in terms of  $N(m)$  which has PMF

$$P_{N(m)}(n) = \begin{cases} (6m)^n e^{-6m}/n! & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

- (a) The probability of no queries in a one minute interval is  $P_{N(1)}(0) = 6^0 e^{-6}/0! = 0.0025$ .
- (b) The probability of exactly 6 queries arriving in a one minute interval is  $P_{N(1)}(6) = 6^6 e^{-6}/6! = 0.1606$ .
- (c) The probability of exactly three queries arriving in a one-half minute interval is  $P_{N(0.5)}(3) = 3^3 e^{-3}/3! = 0.2240$ .

**Problem 10.5.3 •**

At a successful garage, there is always a backlog of cars waiting to be serviced. The service times of cars are iid exponential random variables with a mean service time of 30 minutes. Find the PMF of  $N(t)$ , the number of cars serviced in the first  $t$  hours of the day.

**Problem 10.5.3 Solution**

Since there is always a backlog and the service times are iid exponential random variables, the time between service completions are a sequence of iid exponential random variables. that is, the service completions are a Poisson process. Since the expected service time is 30 minutes, the rate of the Poisson process is  $\lambda = 1/30$  per minute. Since  $t$  hours equals  $60t$  minutes, the expected number serviced is  $\lambda(60t)$  or  $2t$ . Moreover, the number serviced in the first  $t$  hours has the Poisson PMF

$$P_{N(t)}(n) = \begin{cases} \frac{(2t)^n e^{-2t}}{n!} & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

**Problem 10.5.4 ●**

The count of students dropping the course “Probability and Stochastic Processes” is known to be a Poisson process of rate 0.1 drops per day. Starting with day 0, the first day of the semester, let  $D(t)$  denote the number of students that have dropped after  $t$  days. What is  $P_{D(t)}(d)$ ?

**Problem 10.5.4 Solution**

Since  $D(t)$  is a Poisson process with rate 0.1 drops/day, the random variable  $D(t)$  is a Poisson random variable with parameter  $\alpha = 0.1t$ . The PMF of  $D(t)$ , the number of drops after  $t$  days, is

$$P_{D(t)}(d) = \begin{cases} (0.1t)^d e^{-0.1t}/d! & d = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

**Problem 10.5.5 ●**

Customers arrive at the Veryfast Bank as a Poisson process of rate  $\lambda$  customers per minute. Each arriving customer is immediately served by a teller. After being served, each customer immediately leaves the bank. The time a customer spends with a teller is called the service time. If the service time of a customer is exactly two minutes, what is the PMF of the number of customers  $N(t)$  in service at the bank at time  $t$ ?

**Problem 10.5.5 Solution**

Note that it matters whether  $t \geq 2$  minutes. If  $t \leq 2$ , then any customers that have arrived must still be in service. Since a Poisson number of arrivals occur during  $(0, t]$ ,

$$P_{N(t)}(n) = \begin{cases} (\lambda t)^n e^{-\lambda t}/n! & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (0 \leq t \leq 2) \quad (1)$$

For  $t \geq 2$ , the customers in service are precisely those customers that arrived in the interval  $(t-2, t]$ . The number of such customers has a Poisson PMF with mean  $\lambda[t - (t-2)] = 2\lambda$ . The resulting PMF of  $N(t)$  is

$$P_{N(t)}(n) = \begin{cases} (2\lambda)^n e^{-2\lambda}/n! & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (t \geq 2) \quad (2)$$

**Problem 10.5.6 ■**

A sequence of queries are made to a database system. The response time of the system,  $T$  seconds, is an exponential random variable with mean 8. As soon as the system responds to a query, the next query is made. Assuming the first query is made at time zero, let  $N(t)$  denote the number of queries made by time  $t$ .

- What is  $P[T \geq 4]$ , the probability that a single query will last at least four seconds?
- If the database user has been waiting five seconds for a response, what is  $P[T \geq 13 | T \geq 5]$ , the probability that it will last at least eight more seconds?

(c) What is the PMF of  $N(t)$ ?

### Problem 10.5.6 Solution

The time  $T$  between queries are independent exponential random variables with PDF

$$f_T(t) = \begin{cases} (1/8)e^{-t/8} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

From the PDF, we can calculate for  $t > 0$ ,

$$P[T \geq t] = \int_0^t f_T(t') dt' = e^{-t/8} \quad (2)$$

Using this formula, each question can be easily answered.

(a)  $P[T \geq 4] = e^{-4/8} \approx 0.951$ .

(b)

$$P[T \geq 13 | T \geq 5] = \frac{P[T \geq 13, T \geq 5]}{P[T \geq 5]} \quad (3)$$

$$= \frac{P[T \geq 13]}{P[T \geq 5]} = \frac{e^{-13/8}}{e^{-5/8}} = e^{-1} \approx 0.368 \quad (4)$$

(c) Although the time between queries are independent exponential random variables,  $N(t)$  is not exactly a Poisson random process because the first query occurs at time  $t = 0$ . Recall that in a Poisson process, the first arrival occurs some time after  $t = 0$ . However  $N(t) - 1$  is a Poisson process of rate 8. Hence, for  $n = 0, 1, 2, \dots$ ,

$$P[N(t) - 1 = n] = (t/8)^n e^{-t/8} / n! \quad (5)$$

Thus, for  $n = 1, 2, \dots$ , the PMF of  $N(t)$  is

$$P_{N(t)}(n) = P[N(t) - 1 = n - 1] = (t/8)^{n-1} e^{-t/8} / (n - 1)! \quad (6)$$

The complete expression of the PMF of  $N(t)$  is

$$P_{N(t)}(n) = \begin{cases} (t/8)^{n-1} e^{-t/8} / (n - 1)! & n = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (7)$$

### Problem 10.6.1 •

Customers arrive at a casino as a Poisson process of rate 100 customers per hour. Upon arriving, each customer must flip a coin, and only those customers who flip heads actually enter the casino. Let  $N(t)$  denote the process of customers entering the casino. Find the PMF of  $N$ , the number of customers who arrive between 5 PM and 7 PM.

**Problem 10.6.1 Solution**

Customers entering (or not entering) the casino is a Bernoulli decomposition of the Poisson process of arrivals at the casino doors. By Theorem 10.6, customers entering the casino are a Poisson process of rate  $100/2 = 50$  customers/hour. Thus in the two hours from 5 to 7 PM, the number,  $N$ , of customers entering the casino is a Poisson random variable with expected value  $\alpha = 2 \cdot 50 = 100$ . The PMF of  $N$  is

$$P_N(n) = \begin{cases} 100^n e^{-100}/n! & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

**Problem 10.6.3 ♦**

Continuing Problem 10.5.5, suppose each service time is either one minute or two minutes equiprobably, independent of the arrival process or the other service times. What is the PMF of the number of customers  $N(t)$  in service at the bank at time  $t$ ?

**Problem 10.6.3 Solution**

We start with the case when  $t \geq 2$ . When each service time is equally likely to be either 1 minute or 2 minutes, we have the following situation. Let  $M_1$  denote those customers that arrived in the interval  $(t - 1, 1]$ . All  $M_1$  of these customers will be in the bank at time  $t$  and  $M_1$  is a Poisson random variable with mean  $\lambda$ .

Let  $M_2$  denote the number of customers that arrived during  $(t - 2, t - 1]$ . Of course,  $M_2$  is Poisson with expected value  $\lambda$ . We can view each of the  $M_2$  customers as flipping a coin to determine whether to choose a 1 minute or a 2 minute service time. Only those customers that chooses a 2 minute service time will be in service at time  $t$ . Let  $M'_2$  denote those customers choosing a 2 minute service time. It should be clear that  $M'_2$  is a Poisson number of Bernoulli random variables. Theorem 10.6 verifies that using Bernoulli trials to decide whether the arrivals of a rate  $\lambda$  Poisson process should be counted yields a Poisson process of rate  $p\lambda$ . A consequence of this result is that a Poisson number of Bernoulli (success probability  $p$ ) random variables has Poisson PMF with mean  $p\lambda$ . In this case,  $M'_2$  is Poisson with mean  $\lambda/2$ . Moreover, the number of customers in service at time  $t$  is  $N(t) = M_1 + M'_2$ . Since  $M_1$  and  $M'_2$  are independent Poisson random variables, their sum  $N(t)$  also has a Poisson PMF. This was verified in Theorem 6.9. Hence  $N(t)$  is Poisson with mean  $E[N(t)] = E[M_1] + E[M'_2] = 3\lambda/2$ . The PMF of  $N(t)$  is

$$P_{N(t)}(n) = \begin{cases} (3\lambda/2)^n e^{-3\lambda/2}/n! & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (t \geq 2) \quad (1)$$

Now we can consider the special cases arising when  $t < 2$ . When  $0 \leq t < 1$ , every arrival is still in service. Thus the number in service  $N(t)$  equals the number of arrivals and has the PMF

$$P_{N(t)}(n) = \begin{cases} (\lambda t)^n e^{-\lambda t}/n! & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (0 \leq t \leq 1) \quad (2)$$

When  $1 \leq t < 2$ , let  $M_1$  denote the number of customers in the interval  $(t - 1, t]$ . All  $M_1$  customers arriving in that interval will be in service at time  $t$ . The  $M_2$  customers arriving in the interval  $(0, t - 1]$  must each flip a coin to decide one a 1 minute or two

minute service time. Only those customers choosing the two minute service time will be in service at time  $t$ . Since  $M_2$  has a Poisson PMF with mean  $\lambda(t-1)$ , the number  $M'_2$  of those customers in the system at time  $t$  has a Poisson PMF with mean  $\lambda(t-1)/2$ . Finally, the number of customers in service at time  $t$  has a Poisson PMF with expected value  $E[N(t)] = E[M_1] + E[M'_2] = \lambda + \lambda(t-1)/2$ . Hence, the PMF of  $N(t)$  becomes

$$P_{N(t)}(n) = \begin{cases} (\lambda(t+1)/2)^n e^{-\lambda(t+1)/2} / n! & n = 0, 1, 2, \dots \\ 0 & \text{otherwise} \end{cases} \quad (1 \leq t \leq 2) \quad (3)$$

### Problem 10.8.1 ●

$X_n$  is an iid random sequence with mean  $E[X_n] = \mu_X$  and variance  $\text{Var}[X_n] = \sigma_X^2$ . What is the autocovariance  $C_X[m, k]$ ?

### Problem 10.8.1 Solution

The discrete time autocovariance function is

$$C_X[m, k] = E[(X_m - \mu_X)(X_{m+k} - \mu_X)] \quad (1)$$

for  $k = 0$ ,  $C_X[m, 0] = \text{Var}[X_m] = \sigma_X^2$ . For  $k \neq 0$ ,  $X_m$  and  $X_{m+k}$  are independent so that

$$C_X[m, k] = E[(X_m - \mu_X)] E[(X_{m+k} - \mu_X)] = 0(0) = 0 \quad (2)$$

Thus the autocovariance of  $X_n$  is

$$C_X[m, k] = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases} \quad (3)$$

We see that the autocovariance matrix of an independent sequence is diagonal.

### Problem 10.8.2 ■

For the time delayed ramp process  $X(t)$  from Problem 10.3.1, find for any  $t \geq 0$ :

- The expected value function  $\mu_X(t)$
- The autocovariance function  $C_X(t, \tau)$ . Hint:  $E[W] = 1$  and  $E[W^2] = 2$ .

### Problem 10.8.2 Solution

Recall that  $X(t) = t - W$  where  $E[W] = 1$  and  $E[W^2] = 2$ .

- The mean is  $\mu_X(t) = E[t - W] = t - E[W] = t - 1$ .
- The autocovariance is

$$C_X(t, \tau) = E[X(t)X(t+\tau)] - \mu_X(t)\mu_X(t+\tau) \quad (1)$$

$$= E[(t - W)(t + \tau - W)] - (t - 1)(t + \tau - 1) \quad (2)$$

$$= t(t + \tau) - E[(2t + \tau)W] + E[W^2] - t(t + \tau) + 2t + \tau - 1 \quad (3)$$

$$= -(2t + \tau)E[W] + 2 + 2t + \tau - 1 \quad (4)$$

$$= 1 \quad (5)$$

**Problem 10.8.3 ■**

A simple model (in degrees Celsius) for the daily temperature process  $C(t)$  of Example 10.3 is

$$C_n = 16 \left[ 1 - \cos \left( \frac{2\pi n}{365} \right) \right] + 4X_n$$

where  $X_1, X_2, \dots$  is an iid random sequence of Gaussian  $(0, 1)$  random variables.

- What is the mean  $E[C_n]$ ?
- Find the autocovariance function  $C_C[m, k]$ .
- Why is this model overly simple?

**Problem 10.8.3 Solution**

In this problem, the daily temperature process results from

$$C_n = 16 \left[ 1 - \cos \left( \frac{2\pi n}{365} \right) \right] + 4X_n \quad (1)$$

where  $X_n$  is an iid random sequence of  $N[0, 1]$  random variables. The hardest part of this problem is distinguishing between the process  $C_n$  and the covariance function  $C_C[k]$ .

- The expected value of the process is

$$E[C_n] = 16E \left[ 1 - \cos \left( \frac{2\pi n}{365} \right) \right] + 4E[X_n] = 16 \left[ 1 - \cos \left( \frac{2\pi n}{365} \right) \right] \quad (2)$$

*because we are given that  $X_n$  has zero mean.*

- The autocovariance of  $C_n$  is

$$C_C[m, k] = E \left[ \left( C_m - 16 \left[ 1 - \cos \left( \frac{2\pi m}{365} \right) \right] \right) \left( C_{m+k} - 16 \left[ 1 - \cos \left( \frac{2\pi(m+k)}{365} \right) \right] \right) \right] \quad (3)$$

$$= E[4X_m 4X_{m+k}] = 16E[X_m X_{m+k}] = \begin{cases} 16 & k = 0 \\ 0 & \text{otherwise} \end{cases} \quad (4)$$

*because the Gaussian random variable  $X_n$  has variance 1 and the  $X_n$  are independent.*

- A model of this type may be able to capture the mean and variance of the daily temperature. However, one reason this model is overly simple is that *according to this model*, day to day temperatures are uncorrelated. A more realistic model might incorporate the effects of “heat waves” or “cold spells” through correlated daily temperatures.

**Problem 10.9.1 •**

For an arbitrary constant  $a$ , let  $Y(t) = X(t + a)$ . If  $X(t)$  is a stationary random process, is  $Y(t)$  stationary?

**Problem 10.9.1 Solution**

For an arbitrary set of samples  $Y(t_1), \dots, Y(t_k)$ , we observe that  $Y(t_j) = X(t_j + a)$ . This implies

$$f_{Y(t_1), \dots, Y(t_k)}(y_1, \dots, y_k) = f_{X(t_1+a), \dots, X(t_k+a)}(y_1, \dots, y_k) \quad (1)$$

Thus,

$$f_{Y(t_1+\tau), \dots, Y(t_k+\tau)}(y_1, \dots, y_k) = f_{X(t_1+\tau+a), \dots, X(t_k+\tau+a)}(y_1, \dots, y_k) \quad (2)$$

Since  $X(t)$  is a stationary process,

$$f_{X(t_1+\tau+a), \dots, X(t_k+\tau+a)}(y_1, \dots, y_k) = f_{X(t_1+a), \dots, X(t_k+a)}(y_1, \dots, y_k) \quad (3)$$

This implies

$$f_{Y(t_1+\tau), \dots, Y(t_k+\tau)}(y_1, \dots, y_k) = f_{X(t_1+a), \dots, X(t_k+a)}(y_1, \dots, y_k) \quad (4)$$

$$= f_{Y(t_1), \dots, Y(t_k)}(y_1, \dots, y_k) \quad (5)$$

We can conclude that  $Y(t)$  is a stationary process.

**Problem 10.9.2 •**

For an arbitrary constant  $a$ , let  $Y(t) = X(at)$ . If  $X(t)$  is a stationary random process, is  $Y(t)$  stationary?

**Problem 10.9.2 Solution**

For an arbitrary set of samples  $Y(t_1), \dots, Y(t_k)$ , we observe that  $Y(t_j) = X(at_j)$ . This implies

$$f_{Y(t_1), \dots, Y(t_k)}(y_1, \dots, y_k) = f_{X(at_1), \dots, X(at_k)}(y_1, \dots, y_k) \quad (1)$$

Thus,

$$f_{Y(t_1+\tau), \dots, Y(t_k+\tau)}(y_1, \dots, y_k) = f_{X(at_1+a\tau), \dots, X(at_k+a\tau)}(y_1, \dots, y_k) \quad (2)$$

We see that a time offset of  $\tau$  for the  $Y(t)$  process corresponds to an offset of time  $\tau' = a\tau$  for the  $X(t)$  process. Since  $X(t)$  is a stationary process,

$$f_{Y(t_1+\tau), \dots, Y(t_k+\tau)}(y_1, \dots, y_k) = f_{X(at_1+\tau'), \dots, X(at_k+\tau')}(y_1, \dots, y_k) \quad (3)$$

$$= f_{X(at_1), \dots, X(at_k)}(y_1, \dots, y_k) \quad (4)$$

$$= f_{Y(t_1), \dots, Y(t_k)}(y_1, \dots, y_k) \quad (5)$$

We can conclude that  $Y(t)$  is a stationary process.

**Problem 10.9.3 •**

Let  $X(t)$  be a stationary continuous-time random process. By sampling  $X(t)$  every  $\Delta$  seconds, we obtain the discrete-time random sequence  $Y_n = X(n\Delta)$ . Is  $Y_n$  a stationary random sequence?

**Problem 10.9.3 Solution**

For a set of time samples  $n_1, \dots, n_m$  and an offset  $k$ , we note that  $Y_{n_i+k} = X((n_i+k)\Delta)$ . This implies

$$f_{Y_{n_1+k}, \dots, Y_{n_m+k}}(y_1, \dots, y_m) = f_{X((n_1+k)\Delta), \dots, X((n_m+k)\Delta)}(y_1, \dots, y_m) \quad (1)$$

Since  $X(t)$  is a stationary process,

$$f_{X((n_1+k)\Delta), \dots, X((n_m+k)\Delta)}(y_1, \dots, y_m) = f_{X(n_1\Delta), \dots, X(n_m\Delta)}(y_1, \dots, y_m) \quad (2)$$

Since  $X(n_i\Delta) = Y_{n_i}$ , we see that

$$f_{Y_{n_1+k}, \dots, Y_{n_m+k}}(y_1, \dots, y_m) = f_{Y_{n_1}, \dots, Y_{n_m}}(y_1, \dots, y_m) \quad (3)$$

Hence  $Y_n$  is a stationary random sequence.

**Problem 10.9.5 ■**

Let  $A$  be a nonnegative random variable that is independent of any collection of samples  $X(t_1), \dots, X(t_k)$  of a stationary random process  $X(t)$ . Is  $Y(t) = AX(t)$  a stationary random process?

**Problem 10.9.5 Solution**

Given  $A = a$ ,  $Y(t) = aX(t)$  which is a special case of  $Y(t) = aX(t) + b$  given in Theorem 10.10. Applying the result of Theorem 10.10 with  $b = 0$  yields

$$f_{Y(t_1), \dots, Y(t_n)|A}(y_1, \dots, y_n|a) = \frac{1}{a^n} f_{X(t_1), \dots, X(t_n)}\left(\frac{y_1}{a}, \dots, \frac{y_n}{a}\right) \quad (1)$$

Integrating over the PDF  $f_A(a)$  yields

$$f_{Y(t_1), \dots, Y(t_n)}(y_1, \dots, y_n) = \int_0^\infty f_{Y(t_1), \dots, Y(t_n)|A}(y_1, \dots, y_n|a) f_A(a) da \quad (2)$$

$$= \int_0^\infty \frac{1}{a^n} f_{X(t_1), \dots, X(t_n)}\left(\frac{y_1}{a}, \dots, \frac{y_n}{a}\right) f_A(a) da \quad (3)$$

This complicated expression can be used to find the joint PDF of  $Y(t_1 + \tau), \dots, Y(t_n + \tau)$ :

$$f_{Y(t_1+\tau), \dots, Y(t_n+\tau)}(y_1, \dots, y_n) = \int_0^\infty \frac{1}{a^n} f_{X(t_1+\tau), \dots, X(t_n+\tau)}\left(\frac{y_1}{a}, \dots, \frac{y_n}{a}\right) f_A(a) da \quad (4)$$

Since  $X(t)$  is a stationary process, the joint PDF of  $X(t_1 + \tau), \dots, X(t_n + \tau)$  is the same as the joint PDF of  $X(t_1), \dots, X(t_n)$ . Thus

$$f_{Y(t_1+\tau), \dots, Y(t_n+\tau)}(y_1, \dots, y_n) = \int_0^\infty \frac{1}{a^n} f_{X(t_1+\tau), \dots, X(t_n+\tau)}\left(\frac{y_1}{a}, \dots, \frac{y_n}{a}\right) f_A(a) da \quad (5)$$

$$= \int_0^\infty \frac{1}{a^n} f_{X(t_1), \dots, X(t_n)}\left(\frac{y_1}{a}, \dots, \frac{y_n}{a}\right) f_A(a) da \quad (6)$$

$$= f_{Y(t_1), \dots, Y(t_n)}(y_1, \dots, y_n) \quad (7)$$

We can conclude that  $Y(t)$  is a stationary process.

### Problem 10.10.1 •

Which of the following are valid autocorrelation functions?

$$R_1(\tau) = \delta(\tau)$$

$$R_2(\tau) = \delta(\tau) + 10$$

$$R_3(\tau) = \delta(\tau - 10)$$

$$R_4(\tau) = \delta(\tau) - 10$$

### Problem 10.10.1 Solution

The autocorrelation function  $R_X(\tau) = \delta(\tau)$  is mathematically valid in the sense that it meets the conditions required in Theorem 10.12. That is,

$$R_X(\tau) = \delta(\tau) \geq 0 \quad (1)$$

$$R_X(\tau) = \delta(\tau) = \delta(-\tau) = R_X(-\tau) \quad (2)$$

$$R_X(\tau) \leq R_X(0) = \delta(0) \quad (3)$$

However, for a process  $X(t)$  with the autocorrelation  $R_X(\tau) = \delta(\tau)$ , Definition 10.16 says that the average power of the process is

$$E[X^2(t)] = R_X(0) = \delta(0) = \infty \quad (4)$$

Processes with infinite average power cannot exist in practice.

### Problem 10.10.2 •

Let  $A$  be a nonnegative random variable that is independent of any collection of samples  $X(t_1), \dots, X(t_k)$  of a wide sense stationary random process  $X(t)$ . Is  $Y(t) = A + X(t)$  a wide sense stationary process?

### Problem 10.10.2 Solution

To show that  $Y(t)$  is wide-sense stationary we must show that it meets the two requirements of Definition 10.15, namely that its expected value and autocorrelation function must be independent of  $t$ . Since  $Y(t) = A + X(t)$ , the mean of  $Y(t)$  is

$$E[Y(t)] = E[A] + E[X(t)] = E[A] + \mu_X \quad (1)$$

The autocorrelation of  $Y(t)$  is

$$R_Y(t, \tau) = E[(A + X(t))(A + X(t + \tau))] \quad (2)$$

$$= E[A^2] + E[A]E[X(t)] + AE[X(t + \tau)] + E[X(t)X(t + \tau)] \quad (3)$$

$$= E[A^2] + 2E[A]\mu_X + R_X(\tau), \quad (4)$$

where the last equality is justified by the fact that we are given that  $X(t)$  is wide sense stationary. We see that neither  $E[Y(t)]$  nor  $R_Y(t, \tau)$  depend on  $t$ . Thus  $Y(t)$  is a wide sense stationary process.

### Problem 10.10.3 ■

Consider the random process

$$W(t) = X \cos(2\pi f_0 t) + Y \sin(2\pi f_0 t)$$

where  $X$  and  $Y$  are uncorrelated random variables, each with expected value 0 and variance  $\sigma^2$ . Find the autocorrelation  $R_W(t, \tau)$ . Is  $W(t)$  wide sense stationary?

### Problem 10.10.3 Solution

In this problem, we find the autocorrelation  $R_W(t, \tau)$  when

$$W(t) = X \cos 2\pi f_0 t + Y \sin 2\pi f_0 t, \quad (1)$$

and  $X$  and  $Y$  are uncorrelated random variables with  $E[X] = E[Y] = 0$ .

We start by writing

$$R_W(t, \tau) = E[W(t)W(t + \tau)] \quad (2)$$

$$= E[(X \cos 2\pi f_0 t + Y \sin 2\pi f_0 t)(X \cos 2\pi f_0(t + \tau) + Y \sin 2\pi f_0(t + \tau))]. \quad (3)$$

Since  $X$  and  $Y$  are uncorrelated,  $E[XY] = E[X]E[Y] = 0$ . Thus, when we expand  $E[W(t)W(t + \tau)]$  and take the expectation, all of the  $XY$  cross terms will be zero. This implies

$$R_W(t, \tau) = E[X^2] \cos 2\pi f_0 t \cos 2\pi f_0(t + \tau) + E[Y^2] \sin 2\pi f_0 t \sin 2\pi f_0(t + \tau) \quad (4)$$

Since  $E[X] = E[Y] = 0$ ,

$$E[X^2] = \text{Var}[X] - (E[X])^2 = \sigma^2, \quad E[Y^2] = \text{Var}[Y] - (E[Y])^2 = \sigma^2. \quad (5)$$

In addition, from Math Fact B.2, we use the formulas

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)] \quad (6)$$

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)] \quad (7)$$

to write

$$R_W(t, \tau) = \frac{\sigma^2}{2} (\cos 2\pi f_0 \tau + \cos 2\pi f_0 (2t + \tau)) + \frac{\sigma^2}{2} (\cos 2\pi f_0 \tau - \cos 2\pi f_0 (2t + \tau)) \quad (8)$$

$$= \sigma^2 \cos 2\pi f_0 \tau \quad (9)$$

Thus  $R_W(t, \tau) = R_W(\tau)$ . Since

$$E[W(t)] = E[X] \cos 2\pi f_0 t + E[Y] \sin 2\pi f_0 t = 0, \quad (10)$$

we can conclude that  $W(t)$  is a wide sense stationary process. However, we note that if  $E[X^2] \neq E[Y^2]$ , then the  $\cos 2\pi f_0 (2t + \tau)$  terms in  $R_W(t, \tau)$  would not cancel and  $W(t)$  would not be wide sense stationary.

#### Problem 10.10.4 ■

$X(t)$  is a wide sense stationary random process with average power equal to 1. Let  $\Theta$  denote a random variable with uniform distribution over  $[0, 2\pi]$  such that  $X(t)$  and  $\Theta$  are independent.

- What is  $E[X^2(t)]$ ?
- What is  $E[\cos(2\pi f_c t + \Theta)]$ ?
- Let  $Y(t) = X(t) \cos(2\pi f_c t + \Theta)$ . What is  $E[Y(t)]$ ?
- What is the average power of  $Y(t)$ ?

#### Problem 10.10.4 Solution

- In the problem statement, we are told that  $X(t)$  has average power equal to 1. By Definition 10.16, the average power of  $X(t)$  is  $E[X^2(t)] = 1$ .
- Since  $\Theta$  has a uniform PDF over  $[0, 2\pi]$ ,

$$f_{\Theta}(\theta) = \begin{cases} 1/(2\pi) & 0 \leq \theta \leq 2\pi \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

The expected value of the random phase cosine is

$$E[\cos(2\pi f_c t + \Theta)] = \int_{-\infty}^{\infty} \cos(2\pi f_c t + \theta) f_{\Theta}(\theta) d\theta \quad (2)$$

$$= \int_0^{2\pi} \cos(2\pi f_c t + \theta) \frac{1}{2\pi} d\theta \quad (3)$$

$$= \frac{1}{2\pi} \sin(2\pi f_c t + \theta) \Big|_0^{2\pi} \quad (4)$$

$$= \frac{1}{2\pi} (\sin(2\pi f_c t + 2\pi) - \sin(2\pi f_c t)) = 0 \quad (5)$$

(c) Since  $X(t)$  and  $\Theta$  are independent,

$$E[Y(t)] = E[X(t) \cos(2\pi f_c t + \Theta)] = E[X(t)] E[\cos(2\pi f_c t + \Theta)] = 0 \quad (6)$$

Note that the mean of  $Y(t)$  is zero no matter what the mean of  $X(t)$  since the random phase cosine has zero mean.

(d) Independence of  $X(t)$  and  $\Theta$  results in the average power of  $Y(t)$  being

$$E[Y^2(t)] = E[X^2(t) \cos^2(2\pi f_c t + \Theta)] \quad (7)$$

$$= E[X^2(t)] E[\cos^2(2\pi f_c t + \Theta)] \quad (8)$$

$$= E[\cos^2(2\pi f_c t + \Theta)] \quad (9)$$

Note that we have used the fact from part (a) that  $X(t)$  has unity average power. To finish the problem, we use the trigonometric identity  $\cos^2 \phi = (1 + \cos 2\phi)/2$ . This yields

$$E[Y^2(t)] = E\left[\frac{1}{2}(1 + \cos(2\pi(2f_c)t + \Theta))\right] = 1/2 \quad (10)$$

Note that  $E[\cos(2\pi(2f_c)t + \Theta)] = 0$  by the argument given in part (b) with  $2f_c$  replacing  $f_c$ .

### Problem 10.11.1 •

$X(t)$  and  $Y(t)$  are independent wide sense stationary processes with expected values  $\mu_X$  and  $\mu_Y$  and autocorrelation functions  $R_X(\tau)$  and  $R_Y(\tau)$  respectively. Let  $W(t) = X(t)Y(t)$ .

(a) Find  $\mu_W$  and  $R_W(t, \tau)$  and show that  $W(t)$  is wide sense stationary.

(b) Are  $W(t)$  and  $X(t)$  jointly wide sense stationary?

### Problem 10.11.1 Solution

(a) Since  $X(t)$  and  $Y(t)$  are independent processes,

$$E[W(t)] = E[X(t)Y(t)] = E[X(t)] E[Y(t)] = \mu_X \mu_Y. \quad (1)$$

In addition,

$$R_W(t, \tau) = E[W(t)W(t + \tau)] \quad (2)$$

$$= E[X(t)Y(t)X(t + \tau)Y(t + \tau)] \quad (3)$$

$$= E[X(t)X(t + \tau)] E[Y(t)Y(t + \tau)] \quad (4)$$

$$= R_X(\tau)R_Y(\tau) \quad (5)$$

We can conclude that  $W(t)$  is wide sense stationary.

(b) To examine whether  $X(t)$  and  $W(t)$  are jointly wide sense stationary, we calculate

$$R_{WX}(t, \tau) = E[W(t)X(t + \tau)] = E[X(t)Y(t)X(t + \tau)]. \quad (6)$$

By independence of  $X(t)$  and  $Y(t)$ ,

$$R_{WX}(t, \tau) = E[X(t)X(t + \tau)]E[Y(t)] = \mu_Y R_X(\tau). \quad (7)$$

Since  $W(t)$  and  $X(t)$  are both wide sense stationary and since  $R_{WX}(t, \tau)$  depends only on the time difference  $\tau$ , we can conclude from Definition 10.18 that  $W(t)$  and  $X(t)$  are jointly wide sense stationary.

### Problem 10.11.2 ■

$X(t)$  is a wide sense stationary random process. For each process  $X_i(t)$  defined below, determine whether  $X_i(t)$  and  $X(t)$  are jointly wide sense stationary.

(a)  $X_1(t) = X(t + a)$

(b)  $X_2(t) = X(at)$

### Problem 10.11.2 Solution

To show that  $X(t)$  and  $X_i(t)$  are jointly wide sense stationary, we must first show that  $X_i(t)$  is wide sense stationary and then we must show that the cross correlation  $R_{XX_i}(t, \tau)$  is only a function of the time difference  $\tau$ . For each  $X_i(t)$ , we have to check whether these facts are implied by the fact that  $X(t)$  is wide sense stationary.

(a) Since  $E[X_1(t)] = E[X(t + a)] = \mu_X$  and

$$R_{X_1}(t, \tau) = E[X_1(t)X_1(t + \tau)] \quad (1)$$

$$= E[X(t + a)X(t + \tau + a)] \quad (2)$$

$$= R_X(\tau), \quad (3)$$

we have verified that  $X_1(t)$  is wide sense stationary. Now we calculate the cross correlation

$$R_{XX_1}(t, \tau) = E[X(t)X_1(t + \tau)] \quad (4)$$

$$= E[X(t)X(t + \tau + a)] \quad (5)$$

$$= R_X(\tau + a). \quad (6)$$

Since  $R_{XX_1}(t, \tau)$  depends on the time difference  $\tau$  but not on the absolute time  $t$ , we conclude that  $X(t)$  and  $X_1(t)$  are jointly wide sense stationary.

(b) Since  $E[X_2(t)] = E[X(at)] = \mu_X$  and

$$R_{X_2}(t, \tau) = E[X_2(t)X_2(t + \tau)] \quad (7)$$

$$= E[X(at)X(a(t + \tau))] \quad (8)$$

$$= E[X(at)X(at + a\tau)] = R_X(a\tau), \quad (9)$$

we have verified that  $X_2(t)$  is wide sense stationary. Now we calculate the cross correlation

$$R_{XX_2}(t, \tau) = E [X(t)X_2(t + \tau)] \quad (10)$$

$$= E [X(t)X(a(t + \tau))] \quad (11)$$

$$= R_X((a - 1)t + \tau). \quad (12)$$

Except for the trivial case when  $a = 1$  and  $X_2(t) = X(t)$ ,  $R_{XX_2}(t, \tau)$  depends on both the absolute time  $t$  and the time difference  $\tau$ , we conclude that  $X(t)$  and  $X_2(t)$  are not jointly wide sense stationary.

### Problem 11.1.1 •

Let  $X(t)$  denote a wide sense stationary process with  $\mu_X = 0$  and autocorrelation  $R_X(\tau)$ . Let  $Y(t) = 2 + X(t)$ . What is  $R_Y(t, \tau)$ ? Is  $Y(t)$  wide sense stationary?

#### Problem 11.1.1 Solution

For this problem, it is easiest to work with the expectation operator. The mean function of the output is

$$E [Y(t)] = 2 + E [X(t)] = 2 \quad (1)$$

The autocorrelation of the output is

$$R_Y(t, \tau) = E [(2 + X(t)) (2 + X(t + \tau))] \quad (2)$$

$$= E [4 + 2X(t) + 2X(t + \tau) + X(t)X(t + \tau)] \quad (3)$$

$$= 4 + 2E [X(t)] + 2E [X(t + \tau)] + E [X(t)X(t + \tau)] \quad (4)$$

$$= 4 + R_X(\tau) \quad (5)$$

We see that  $R_Y(t, \tau)$  only depends on the time difference  $\tau$ . Thus  $Y(t)$  is wide sense stationary.

### Problem 11.1.2 •

$X(t)$ , the input to a linear time-invariant filter is a wide sense stationary random process with expected value  $\mu_X = -3$  volts. The impulse response of the filter is

$$h(t) = \begin{cases} 1 - 10^6 t^2 & 0 \leq t \leq 10^{-3} \text{ sec,} \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected value of the output process  $Y(t)$ ?

**Problem 11.1.2 Solution**

By Theorem 11.2, the mean of the output is

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t) dt \quad (1)$$

$$= -3 \int_0^{10^{-3}} (1 - 10^6 t^2) dt \quad (2)$$

$$= -3 \left( t - (10^6/3)t^3 \right) \Big|_0^{10^{-3}} \quad (3)$$

$$= -2 \times 10^{-3} \text{ volts} \quad (4)$$

**Problem 11.1.3 •**

$X(t)$ , the input to a linear time-invariant filter is a wide sense stationary stochastic process with expected value  $\mu_X = 4$  volts. The filter output  $Y(t)$  is a wide sense stationary stochastic process with expected  $\mu_Y = 1$  volt. The filter impulse response is

$$h(t) = \begin{cases} e^{-t/a} & t \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

What is the value of the time constant  $a$ ?

**Problem 11.1.3 Solution**

By Theorem 11.2, the mean of the output is

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t) dt = 4 \int_0^{\infty} e^{-t/a} dt = -4ae^{-t/a} \Big|_0^{\infty} = 4a. \quad (1)$$

Since  $\mu_Y = 1 = 4a$ , we must have  $a = 1/4$ .

**Problem 11.1.4 ■**

A white Gaussian noise signal  $W(t)$  with autocorrelation function  $R_W(\tau) = \eta_0 \delta(\tau)$  is passed through an LTI filter  $h(t)$ . Prove that the output  $Y(t)$  has average power

$$E[Y^2(t)] = \eta_0 \int_{-\infty}^{\infty} h^2(u) du.$$

**Problem 11.1.4 Solution**

Since  $E[Y^2(t)] = R_Y(0)$ , we use Theorem 11.2(a) to evaluate  $R_Y(\tau)$  at  $\tau = 0$ . That is,

$$R_Y(0) = \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v) R_X(u-v) dv du \quad (1)$$

$$= \int_{-\infty}^{\infty} h(u) \int_{-\infty}^{\infty} h(v) \eta_0 \delta(u-v) dv du \quad (2)$$

$$= \eta_0 \int_{-\infty}^{\infty} h^2(u) du, \quad (3)$$

by the sifting property of the delta function.

**Problem 11.5.1 ●**

$X(t)$  is a wide sense stationary process with autocorrelation function

$$R_X(\tau) = 10 \frac{\sin(2000\pi\tau) + \sin(1000\pi\tau)}{2000\pi\tau}.$$

What is the power spectral density of  $X(t)$ ?

**Problem 11.5.1 Solution**

To use Table 11.1, we write  $R_X(\tau)$  in terms of the autocorrelation

$$\text{sinc}(x) = \frac{\sin(\pi x)}{\pi x}. \quad (1)$$

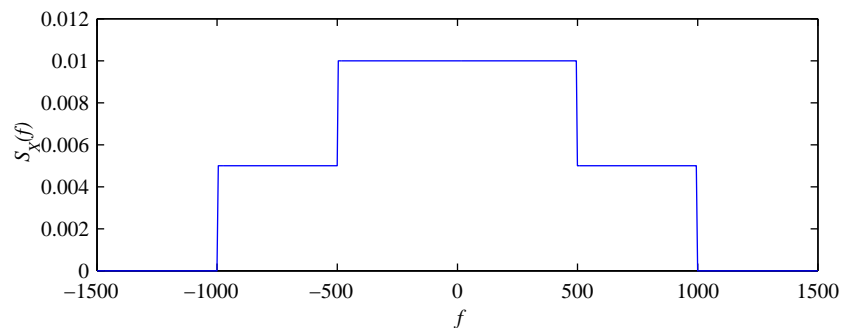
In terms of the  $\text{sinc}(\cdot)$  function, we obtain

$$R_X(\tau) = 10 \text{sinc}(2000\tau) + 5 \text{sinc}(1000\tau). \quad (2)$$

From Table 11.1,

$$S_X(f) = \frac{10}{2,000} \text{rect}\left(\frac{f}{2000}\right) + \frac{5}{1,000} \text{rect}\left(\frac{f}{1,000}\right) \quad (3)$$

Here is a graph of the PSD.

**Problem 11.5.2 ■**

$X(t)$  is a wide sense stationary process with  $\mu_X = 0$  and  $Y(t) = X(\alpha t)$  where  $\alpha$  is a nonzero constant. Find  $R_Y(\tau)$  in terms of  $R_X(\tau)$ . Is  $Y(t)$  wide sense stationary? If so, find the power spectral density  $S_Y(f)$ .

**Problem 11.5.2 Solution**

The process  $Y(t)$  has expected value  $E[Y(t)] = 0$ . The autocorrelation of  $Y(t)$  is

$$R_Y(t, \tau) = E[Y(t)Y(t + \tau)] = E[X(\alpha t)X(\alpha(t + \tau))] = R_X(\alpha\tau) \quad (1)$$

Thus  $Y(t)$  is wide sense stationary. The power spectral density is

$$S_Y(f) = \int_{-\infty}^{\infty} R_X(\alpha\tau) e^{-j2\pi f\tau} d\tau. \quad (2)$$

At this point, we consider the cases  $\alpha > 0$  and  $\alpha < 0$  separately. For  $\alpha > 0$ , the substitution  $\tau' = \alpha\tau$  yields

$$S_Y(f) = \frac{1}{\alpha} \int_{-\infty}^{\infty} R_X(\tau') e^{-j2\pi(f/\alpha)\tau'} d\tau' = \frac{S_X(f/\alpha)}{\alpha} \quad (3)$$

When  $\alpha < 0$ , we start with Equation (2) and make the substitution  $\tau' = -\alpha\tau$ , yielding

$$S_Y(f) = \frac{1}{-\alpha} \int_{-\infty}^{\infty} R_X(-\tau') e^{-j2\pi\frac{f}{-\alpha}\tau'} d\tau'. \quad (4)$$

Since  $R_X(-\tau') = R_X(\tau')$ ,

$$S_Y(f) = \frac{1}{-\alpha} \int_{-\infty}^{\infty} R_X(\tau') e^{-j2\pi\frac{f}{-\alpha}\tau'} d\tau' = \frac{1}{-\alpha} S_X\left(\frac{f}{-\alpha}\right) \quad (5)$$

For  $-\alpha = |\alpha|$  for  $\alpha < 0$ , we can combine the  $\alpha > 0$  and  $\alpha < 0$  cases in the expression

$$S_Y(f) = \frac{1}{|\alpha|} S_X\left(\frac{f}{|\alpha|}\right). \quad (6)$$