Solution to Homework Assignment 4

1. Consider the open-loop transfer function

\[ G(s) = \frac{K(s + 1)}{(s - 1)(s^2 + 8s + 32)}. \]  

(a) Use the Routh array to determine the range of values of \( K \) for which the closed-loop system with negative unity feedback is stable.

\[ p(s) = (s - 1)(s^2 + 8s + 32) + K(s + 1) = s^3 + 7s^2 + (24 + K)s + (K - 32). \]

The Routh array is then

\[
\begin{array}{c|ccc}
  s^3 & 1 & (24 + K) \\
  s^2 & 7 & (K - 32) \\
  s^1 & \alpha & 0 \\
  s^0 & (K - 32)
\end{array}
\]

where

\[ \alpha = \frac{-1}{6} (K - 32 - 7(24 + K)) = \frac{1}{6} (6K + 200) \]

and the first element in the \( s^0 \) row can be computed easily by observing that

\[ \frac{1}{\alpha} (0 - \alpha(K - 32)) = K - 32. \]

Applying the Routh-Hurwitz criterion, the closed loop system is stable if \( K > 32 \) (from the \( s^0 \) row) and \( K > -200/6 \) (from the \( s^1 \) row), so we need \( K > 32 \).

(b) Now select a value of \( K \) that produces the least overshoot and plot the step response of the closed-loop system. Be sure to use a time range that shows the important aspects of the behavior.

**Solution:** Using the Matlab script below, we find that the value of \( \zeta \) seems to decrease with increasing \( K \), which would indicate that to decrease the overshoot we should pick a small \( K \). Of course, we have to figure out what value of “small” is appropriate. In the matlab script, I tested the value \( K = 1 \), and since that yields no overshoot, we can’t do better than that, so \( K = 1 \) will do. The plots are shown in Figures 1 and 2. Here’s the script.

```matlab
%%
%% Homework 4, Problem 1 Solution sk 10/29/08
%%

for index = 1:1000,
    rs(index,:) = roots([1 7 (24 + 32+index) index]);
```
z(index) = -real(rs(index,1))/abs(rs(index,1));
end;
figure(1)
plot(z);
title('HW4 P1: Damping Coefficient $\zeta$ vs. Gain $K$')
xlabel('Gain $K$')
ylabel('Damping Coefficient $\zeta$')
grid
print -depsc 'HW4_p1_fig1'
[y,t] = step(tf([1 1],[1 7 25 1]));
figure(2)
plot(t,y);
title('HW4 P1: Step Response for $K=1$')
xlabel('Time $t$')
ylabel('Output $y(t)$')
grid
print -depsc 'HW4_p1_fig2'
HW4 P1: Step Response for K=1

Output \( y(t) \) vs. Time \( t \)
2. Consider a unity negative feedback system with forward path (open loop) transfer function
\[ G(s) = \frac{K}{s((s + 3 + K/3)(s + p) + (K/3))}. \]  
(a) Draw a block diagram, in which each block contains a transfer function having at most one pole and at most one zero that would implement this closed-loop system. (There is more than one way to do this. Any correct answer will do.)

(b) Assume that the value \( p \) may take any value in the set \( \{1, 2, 3\} \). Find the range of values for the gain \( K \) that result in a stable closed-loop system regardless of which of the three values \( p \) takes.

**Solution:** A little tedious algebra yields the Routh array:

\[
\begin{align*}
s^3 : & \quad 1 \quad \left(3 + \frac{K}{3}\right)p + \frac{K}{3} \\
s^2 : & \quad \frac{9+3p+K}{3} \quad K \\
s^1 : & \quad \frac{9+3p-2K}{9+3p+K} \quad 0 \\
s^0 : & \quad \frac{K}{K}
\end{align*}
\]

which gives us three constraints on the value of \( K \) for the system to be stable, namely

\[
\begin{align*}
9 + 3p + K & > 0 \quad (8) \\
9 + 3p-2K & > 0 \quad (9) \\
K & > 0. \quad (10)
\end{align*}
\]

The first condition simplifies to \( K > -9 - 3p \). For the three given values of \( p \), the maximum of the right hand side occurs when \( p = 1 \), yielding \( K > -12 \). The second condition simplifies to \( K < (9 + 3p)/2 \). The minimum of the right hand side occurs when \( p = 1 \), yielding the constraint \( K < 6 \). Thus, the acceptable range of values for the gain \( K \) is \( 0 < K < 6 \).

3. Consider the Single-Input Single-Output (SISO) system with state space representation
\[
\begin{align*}
\dot{x}(t) & = \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) & = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)
\end{align*}
\]  
and input \( u(t) = -Kx(t) + r(t) \).

(a) Determine the closed loop transfer function \( y(s)/r(s) \).

**Solution:** First, we substitute for \( u \) in the state equation to obtain
\[
\begin{align*}
\dot{x}(t) & = A x(t) + B (-Kx(t) + r(t)) \\
& = (A - BK)x(t) + Br(t) \\
& = \begin{bmatrix} 0 & 2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \\
& = \begin{bmatrix} 0 \\ 1 - k_1 -3 - k_2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r(t) \\
& = A_{cl} x(t) + Br(t).
\end{align*}
\]
The transfer function is then
\[
g(s) = \frac{y(s)}{r(s)} = C(sI - Acl)^{-1}B + D
\]
\[= \left[ \begin{array}{cc} 1 & 0 \\ -1 + k_1 & s + 3 + k_2 \end{array} \right]^{-1} \left[ \begin{array}{c} 0 \\ 1 \end{array} \right] + 0 \]
\[= \frac{2}{s(s + 3 + k_2) - (2(1 - k_1))} = \frac{2}{s^2 + (3 + k_2)s + 2(k_1 - 1)}. \]

(b) Determine the range of values \(k_1\) and \(k_2\) for the closed-loop transfer function to be stable.

**Solution:** Using the Routh Array or any other valid method we find that we need both \(3 + k_2\) and \(2(k_1 - 1)\) positive, or \(k_2 > -3\) and \(k_1 > 1\).

(c) Select values of \(k_1\) and \(k_2\) such that the settling time of the unit step response is 1 second.

**Solution:** The problem should have specified which settling time was required, but we know that if this is not specified, the 2\%-settling time is usually a reasonable choice. Then we want \(1 = T_s = \frac{4}{\omega_n\zeta}\), so we need \(\omega_n\zeta = 4\) and the \(k_i\) must satisfy
\[
\begin{align*}
3 + k_2 &= 2\omega_n\zeta = 8 \\
2(k_1 - 1) &= \omega_n^2.
\end{align*}
\]
Obviously we need \(k_2 = 5\). We have to think a little more to choose an appropriate value for \(k_1\). It might appear that we could choose any \(\omega_n\) whatsoever, but for a step response we would generally want to use a \(\zeta\) around 0.707. To get a quick order of magnitude estimate, I observe that for \(\zeta = 1\) we’d need \(k_1 = 9\). Decreasing \(\zeta\) would increase \(\omega_n\) and thus the required value for \(k_1\). Since 0.707 = \(1/\sqrt{2}\), decreasing \(\zeta\) to 90.707 would increase \(\omega_n\) by \(\sqrt{2}\), hence \(\omega_n^2\) by 4. We thus need to roughly double \(k_1\). To be precise, for \(\zeta = 1/\sqrt{2}\) we need \(k_1 = 17\).

(d) For your matrix \(K\), calculate the step response of the state space system. (That means you need to determine both \(x(t)\) and \(y(t)\)).

**Solution:** Having been given no initial condition we assume \(x(0) = 0\). We note that the output is just \(y = x_2\) so if we determine the step response of the states we will also have the step response of the output. I used the Matlab script below to obtain the step response that follows.

```matlab
%%
%% Homework 4, Problem 3 Solution   sk 11/15/08
%%

k1 = (4/0.707)^2/2+1
k2 = 5;
A = [0 2;1-k1 -3-k2];
B = [0;1];
C = eye(2);
D = 0*B;
```
[y,t] = step(ss(A,B,C,D));
figure(2)
plot(t,y);
title('HW4 P3: Step Response for K=1')
xlabel('Time t')
ylabel('States')
legend('x_1','x_2')
grid
print -depsc 'HW4_p3_fig1'