# Derivation of (8.28) of Dorf and Bishop

The reason that I could not derive that last term of (8.28) in class is that it is wrong in the textbook. What I wrote in class was probably correct but I'll rewrite it here just in case.

We start with (8.26) which was

$$G(j\omega) = \frac{K_b \prod_{i=1}^{Q} (1 + j\omega\tau_i)}{(j\omega)^N \prod_{m=1}^{M} (1 + j\omega\tau_m) \prod_{k=1}^{R} \left[ 1 + \left( \frac{2\zeta_k}{\omega_{n_k}} \right) j\omega + \left( \frac{j\omega}{\omega_{n_k}} \right)^2 \right]}. \tag{8.26}$$

We recall that for products of complex numbers,

$$\angle \left( (a + jb)(c + jd) \right) = \angle \left( e^{j\phi_1} e^{j\phi_2} \right) = \angle \left( e^{j(\phi_1 + \phi_2)} \right) = \phi_1 + \phi_2$$

where $\phi_1 = \tan^{-1}(b/a)$ and $\phi_2 = \tan^{-1}(d/c)$. Accordingly, when we want to write $G(j\omega)$ in the form

$$G(j\omega) = |G(\omega)| \, e^{j\phi(\omega)} \tag{8.16}$$

we can consider individually the factors of $G(j\omega)$ in (8.26) and sum their angle contributions to obtain $\phi(\omega)$.

From the numerator of (8.26) we obtain the following contributions:

$$\begin{aligned} \angle K_b + \sum_{i=1}^{Q} \angle \left( 1 + j\omega\tau_i \right) &= \tan^{-1} \left( \frac{\text{Im } K_b}{\text{Re } K_b} \right) + \sum_{i=1}^{Q} \tan^{-1} \left( \frac{\text{Im } (1 + j\omega\tau_i)}{\text{Re } (1 + j\omega\tau_i)} \right) \\ &= 0 + \sum_{i=1}^{Q} \tan^{-1} \left( \frac{\omega\tau_i}{1} \right) = \sum_{i=1}^{Q} \tan^{-1} \left( \omega\tau_i \right). \end{aligned}$$

This matches the first term of (8.28) of the text.

From the denominator of (8.26) we obtain the following contributions: for the poles at the origin we have the contribution

$$-\angle (j\omega)^N = -N \tan^{-1} \left( \frac{\text{Im } (j\omega)}{\text{Re } (j\omega)} \right) = -N(90 \,\text{deg}),$$

from the poles on the real axis we have

$$-\sum_{m=1}^{M} \angle (1 + j\omega\tau_m) = -\sum_{m=1}^{M} \tan^{-1} \left( \frac{\text{Im } (1 + j\omega\tau_m)}{\text{Re } (1 + j\omega\tau_m)} \right) = -\sum_{m=1}^{M} \tan^{-1} \left( \omega\tau_m \right),$$

and from the complex conjugate pole pairs we have

$$
\begin{aligned}
-\sum_{k=1}^{R} \angle \left[1 + \left(\frac{2\zeta_k}{\omega_{n_k}}\right) j\omega + \left(\frac{j\omega}{\omega_{n_k}}\right)^2\right] &= -\sum_{k=1}^{R} \tan^{-1}\left(\frac{\mathrm{Im}\left[1 + \left(\frac{2\zeta_k}{\omega_{n_k}}\right) j\omega + \left(\frac{j\omega}{\omega_{n_k}}\right)^2\right]}{\mathrm{Re}\left[1 + \left(\frac{2\zeta_k}{\omega_{n_k}}\right) j\omega + \left(\frac{j\omega}{\omega_{n_k}}\right)^2\right]}\right) \\
&= -\sum_{k=1}^{R} \tan^{-1}\left(\frac{\left(\frac{2\zeta_k\omega}{\omega_{n_k}}\right)}{\left(\frac{\omega_{n_k}^2-\omega^2}{\omega_{n_k}^2}\right)}\right) \\
&= -\sum_{k=1}^{R} \tan^{-1}\left(\frac{2\zeta_k\omega_{n_k}\omega}{\omega_{n_k}^2-\omega^2}\right).
\end{aligned}
$$

Thus we see that (8.28) should be

$$
\phi(\omega) = \sum_{i=1}^{Q} \tan^{-1}(\omega\tau_i) - N(90\,\mathrm{deg}) - \sum_{m=1}^{M} \tan^{-1}(\omega\tau_m) - \sum_{k=1}^{R} \tan^{-1}\left(\frac{2\zeta_k\omega_{n_k}\omega}{\omega_{n_k}^2-\omega^2}\right). \quad (8.28)
$$