Discretization of Continuous Time State Space Systems

Suppose we are given the continuous time state space system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t) \quad (1)\\ y(t) &= Cx(t) + Du(t) \quad (2)\end{aligned}$$

and apply an input that changes only at discrete (equal) sampling intervals. It would be nice if we could find matrices G and H, independent of t or k so that we could obtain a discrete time model of the system,

$$\begin{aligned}x\left((k+1)T\right) &= G(T)x(kT) + H(T)u(kT) \quad (3)\\ y(kT) &= Cx(kT) + Du(kT). \quad (4)\end{aligned}$$

We will now determine the values of the matrices $G(T)$ and $H(T)$. It will turn out that while they are constant for a particular sampling interval, they depend on the value of the sampling interval, so for that reason I have written them as $G(T)$ and $H(T)$ in (3) above.

We start by using the solution of (1) to calculate the values of the state x at times kT and $(k+1)T$. These are

$$\begin{aligned}x\left((k+1)T\right) &= e^{A(k+1)T}x(0) + e^{A(k+1)T}\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau \quad (5)\\ x(kT) &= e^{AkT}x(0) + e^{AkT}\int_0^{kT} e^{-A\tau}Bu(\tau)d\tau. \quad (6)\end{aligned}$$

We want to write $x((k+1)T)$ in terms of $x(kT)$ so we multiply all terms of (6) by e^{AT} and solve for $e^{A(k+1)T}x(0)$, obtaining

$$e^{A(k+1)T}x(0) = e^{AT}x(kT) - e^{A(k+1)T}\int_0^{kT} e^{-A\tau}Bu(\tau)d\tau. \quad (7)$$

Substituting for $e^{A(k+1)T}x(0)$ in (5) we obtain

$$x\left((k+1)T\right) = e^{AT}x(kT) + e^{A(k+1)T}\left[\int_0^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau - \int_0^{kT} e^{-A\tau}Bu(\tau)d\tau\right] \quad (8)$$

which, by linearity of integration, is equivalent to

$$x\left((k+1)T\right) = e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}Bu(\tau)d\tau. \quad (9)$$

Next, we notice that within the interval from kT to $(k+1)T$, $u(t) = u(kT)$ is constant, as is the matrix B, so we can take them out of the integral to obtain

$$x\left((k+1)T\right) = e^{AT}x(kT) + e^{A(k+1)T}\int_{kT}^{(k+1)T} e^{-A\tau}d\tau Bu(kT) \qquad \tau \in [kT, (k+1)T). \quad (10)$$

We can take the $e^{A(k+1)T}$ inside the integral to obtain

$$x\left((k+1)T\right) = e^{AT}x(kT) + \int_{kT}^{(k+1)T} e^{A[(k+1)T-\tau]}d\tau Bu(kT) \qquad \tau \in [kT, (k+1)T). \quad (11)$$

Now we see that as τ ranges from kT to $(k+1)T$ (the lower to the upper limit of integration) the exponent of e ranges from T to 0. Accordingly, let's define a new variable $\lambda = (k+1)T - \tau$. Then $d\lambda = -d\tau$ and λ ranges from T to 0 as τ ranges from kT to $(k+1)T$. Thus we have

$$x\left((k+1)T\right) = e^{AT}x(kT) - \int_T^0 e^{A\lambda}d\lambda Bu(kT) \qquad \lambda \in [0, kT), \tag{12}$$

or

$$x\left((k+1)T\right) = e^{AT}x(kT) + \int_0^T e^{A\lambda}d\lambda Bu(kT) \qquad \lambda \in [0, kT). \tag{13}$$

We see that in (13) we have written the state update equation exactly in the form of (3) where

$$G(T) = e^{AT} \tag{14}$$

$$H(T) = \int_0^T e^{A\lambda}d\lambda B, \tag{15}$$

so we're done ...except that we'd rather not leave the expression for $H(T)$ in the form of an integral. So long as A is invertible, we can easily integrate, using the fact that

$$\frac{d}{dt}e^{AT} = Ae^{AT} = e^{AT}A \tag{16}$$

to obtain

$$H(T) = A^{-1}\int_0^T Ae^{A\lambda}d\lambda B = A^{-1}e^{A\lambda}|_{\lambda=0}^T B \tag{17}$$

$$= A^{-1}(e^{AT} - I)B = (e^{AT} - I)BA^{-1}. \tag{18}$$

Finally, note that while I restricted the value of τ and λ to lie within a single sampling interval, k appears nowhere in the expressions for $G(T)$ and $H(T)$. Our solution to (3) is thus

$$x(kT) = (G(T))^k x(0) + \sum_{j=0}^{k-1}(G(T))^{k-j-1}H(T)u(jT), \qquad k = 1, 2, 3, \ldots \tag{19}$$

and we can see that at the sampling instants kT, this has exactly the same value as is obtained using (1). Specifically,

$$(G(T))^k = (e^{AT})^k = e^{AkT}, \tag{20}$$

and since the input $u(t)$ is constant on sampling intervals,

$$e^{AkT}\int_0^{kT} e^{-A\tau}Bu(\tau)d\tau = \sum_{j=0}^{k-1} e^{A(k-j-1)T}A^{-1}(e^{AT} - I)Bu(jT) = \sum_{j=0}^{k-1}(G(T))^{k-j-1}H(T)u(jT). \tag{21}$$