Name: SOLUTION  

Score: /100

You must show ALL of your work, and justify your answers, for full credit.

This exam is closed-book. Calculators may NOT be used. Please leave fractions as fractions, etc. I do not want the decimal equivalents. Please check your work carefully.

1. (15 points) Consider the system

\[
\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u
\]

\[
y = \begin{bmatrix} 1 & 0 \end{bmatrix} x.
\]

(a) Show that the system is not controllable.

The controllability matrix

\[
C = [B \ AB] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} 0 & b_1 \\ b_2 & b_2 \end{bmatrix}
\]

is singular so the system is not controllable.

(b) Can you find a new \(B\) matrix that would make the system controllable? No.

Let \(B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}\). Then

\[
C = [B \ AB] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \begin{bmatrix} 0 & b_1 \\ b_2 & b_2 \end{bmatrix}
\]

which is singular regardless of the values of \(b_1\) and \(b_2\).

(c) Find a matrix \(P\) to transform the system into Kalman controller form and determine the resulting \(\bar{A}\) and \(\bar{B}\) matrices, partitioned to indicate the distinctions between controllable and uncontrollable parts.

The system is already in Kalman controller form. (so \(P = I\))
2. (10 points) Calculate $W_c(t)$ for the (same) continuous-time system

\[
\dot{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u
\]

\[
y = [1 \ 0] x.
\]

Is your answer consistent with your answers to the previous problem?

\[
W_c(t) = \int_0^t e^{A\tau} B B^T e^{A^T \tau} \, d\tau
\]

\[
= \int_0^t \begin{bmatrix} e^{\tau} & 0 \\ 0 & e^{\tau} \end{bmatrix} [1 \ 0] [1 \ 0] [e^{\tau} & 0; 0 \ e^{\tau}] \, d\tau
\]

\[
= \int_0^t [e^\tau \ 0; 0 \ e^\tau] \, d\tau =
\]

\[
= \int_0^t \begin{bmatrix} e^{2\tau} & 0 \\ 0 & 0 \end{bmatrix} \, d\tau = \left[ \frac{1}{2} (e^{2\tau} - 1), 0 \right] 0
\]

which is singular, as it must be since the system is uncontrollable.
3. (10 points) Consider the (new) continuous-time system

\[ \dot{x} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u. \]  

Assume that \( W_c(1) = I \). (It doesn't, but I want to simplify the calculations.) State the appropriate formula and find \( u(t) \) that takes the system from the initial condition \( x(0) = [0 \ 0]^T \) to the final state \( x(1) = [1 \ 1]^T \).

\[
\begin{align*}
    u(t) &= -B^T e^{A^T (1-t)} W_c^{-1}(1) \left( e^{A} x_0 - x_1 \right) \\
    &= - \begin{bmatrix} 1 & 0 \\ (1-t)e^{-(t)} & e^{-(t)} \end{bmatrix} \begin{bmatrix} e^{-(t)} & 0 \\ 0 & e^{-(t)} \end{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \\
    &= \begin{bmatrix} 1 & 0 \\ (2-t)e^{-(t)} & e^{-(t)} \end{bmatrix} \\
    &= e^{-(t)} \quad t > 0
\end{align*}
\]
4. (10 points) Consider the discrete-time system

\[ x[k+1] = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k] \]  

(6)

\[ y[k] = [1 \ 0] x[k]. \]  

(7)

(a) Is the system BIBO stable?

The system is asymptotically stable (see below) hence also BIBO stable.

(b) Is the system marginally stable?

The system is asymptotically stable (see below) so marginally stable.

(c) Is the system asymptotically stable?

Both eigenvalues \( \frac{1}{2} \) and \( \frac{1}{3} \) have magnitude \( < 1 \) so the system is asymptotically stable.
5. (15 points) Consider the (same) discrete-time system
\[
x[k+1] = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[k]
\]
\[
y[k] = [1 \ 0] x[k].
\]

Find a sequence \( u[k] \) that takes the system state from \( x[0] = 0 \) to \( x[3] = [1 \ 2]^T \).

**Solution using formula for \( x[k] \) is on next page.**

\[
x[3] = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} x[2] + \begin{bmatrix} 1 \end{bmatrix} u[2]
\]
\[
= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \left( \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} x[1] + \begin{bmatrix} 1 \end{bmatrix} u[1] \right) + \begin{bmatrix} 1 \end{bmatrix} u[2]
\]
\[
= \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \left( \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \left( \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} x[0] + \begin{bmatrix} 1 \end{bmatrix} u[0] \right) + \begin{bmatrix} 1 \end{bmatrix} u[1] \right) + \begin{bmatrix} 1 \end{bmatrix} u[2]
\]

So with \( x[0] = 0 \),

\[
\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/4 u[0] + \frac{1}{2} u[1] + u[2] \\ 1/4 u[0] + \frac{1}{3} u[1] + u[2] \end{bmatrix}
\]

Since we have 2 equations in three unknowns, one value is arbitrary, so let's take \( u[0] = 0 \). Then

\[1 = \frac{1}{2} u[1] + u[2]\]

Subtracting the first from the 2nd yields

\[2 = \frac{1}{3} u[1] + u[2]\]

\[1 = -\frac{1}{6} u[1] \quad \text{so} \quad u[1] = -6\]

Infinitely, and thus \( u[2] = 4 \). (Many other solutions exist.)
\[ x[k] = A^k x[0] + \sum_{m=0}^{k-1} A^{k-1-m} B u[m] \]

So we need to find \( u[0], u[1], \) and \( u[2] \) s.t.

\[
\begin{bmatrix}
1 \\
2
\end{bmatrix} = 0 + A^2 B u[0] + A B u[1] + B u[2]
\]

\[
= \begin{bmatrix}
\left(\frac{1}{2}\right)^2 u[0] + \frac{1}{2} u[1] + u[2] \\
\left(\frac{1}{3}\right)^2 u[0] + \frac{1}{3} u[1] + u[2]
\end{bmatrix}
\]

Since we have only two equations in 3 unknowns, let's see if we can find a solution with \( u[0] = 0 \).

Then

\[
1 = \frac{1}{2} u[1] + u[2]
\]

\[
2 = \frac{1}{3} u[1] + u[2]
\]

So subtracting the first from the second yields

\[
1 = -\frac{u[1]}{6}
\]

or \( u[1] = -6 \), then from the first \( u[2] = 1 + 3 = 4 \)

Check:

\[
\begin{bmatrix}
1 \\
2
\end{bmatrix} = \begin{bmatrix}
0 - 3 + 4 \\
0 - 2 + 4
\end{bmatrix}
\]
6. (10 points) Consider the continuous-time system

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u. \quad (10)$$

Does the Lyapunov equation $A^T M + M A = -I$ have a solution? Is it unique? Find the solution (if it's not unique, give a parametrized solution.)

Because the eigenvalues are in the LHP and $I$ is symmetric positive definite, there is a solution $M$ which is positive definite.

$$\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

(1,1) \(-m_{11} - m_{11} = -1 \Rightarrow m_{11} = \frac{1}{2}\)

(1,2) \(-m_{12} - 2m_{12} = 0 \Rightarrow m_{12} = 0\)

(2,1) \(-2m_{21} - m_{21} = 0 \Rightarrow m_{21} = 0\)

(2,2) \(-2m_{22} - 2m_{22} = -1 \Rightarrow m_{22} = \frac{1}{4}\)

So $M = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{bmatrix}$ which is, obviously, unique and, having positive eigenvalues $\frac{1}{2}$ and $\frac{1}{4}$, positive definite.
7. (15 points) Give the definitions and conditions for

(a) BIBO stability (give both time-domain and frequency-domain conditions),
A SYSTEM IS BIBO STABLE IF ANY BOUNDED INPUT LEADS TO A BOUNDED OUTPUT IN CONT. TIME, THE TIME DOMAIN CONDITION IS ABSOLUTE INTEGRABILITY, I.E., \( \int_{-\infty}^{\infty} |g(t)|dt < M < \infty \) FOR SOME FINITE M.
THE FREQUENCY DOMAIN CONDITION IS THAT IF \( \hat{g}(\omega) \) IS A PROPER RATIONAL FUNCTION, ITS POLES MUST LIE IN THE LHP.
IN DISCRETE TIME, THE TIME DOMAIN CONDITION IS ABSOLUTE SUMMABILITY, I.E., \( \sum_{k=0}^{\infty} |g[k]| \leq M < \infty \), OF THE impulse response.

(b) Marginal stability, and
A ZIR OF A SYSTEM IS MARGINALLY STABLE IF FOR ANY FINITE I.C. THE RESPONSE IS BOUNDED.
C.T. CONDITION: ALL EIGENVALUES OF \( A \) (THEREAL PART IS) HAVE REAL PART \( \leq 0 \) AND \( \Re(\lambda) < 0 \), THE EIGENVALUE MUST BE A SIMPLE ROOT OF THE MINIMAL POLYNOMIAL. D.T. CONDITION REPLACE "REAL PART \( \leq 0 \)" BY "MAGNITUDE \( \leq 1 \)"

(c) Asymptotic stability.
A SYSTEM IS ASYMPTOTICALLY STABLE IF ANY FINITE INITIAL STATE EXCITES A BOUNDED ZIR THAT GOES TO ZERO AS \( t \to \infty \)
IN C.T. ALL EIGENVALUES MUST HAVE (STRICTLY) NEGATIVE REAL PART. IN C.T. THEY MUST ALL HAVE MAGNITUDE (STRICTLY) LESS THAN ONE.

* RESPONSE FUNCTION, WITH M A FINITE CONSTANT AND IN THE FREQUENCY DOMAIN ALL POLES OF A RATIONAL PROPER TRANSFER FUNCTION SHOULD HAVE MAGNITUDE LESS THAN ONE.
8. (15 points) Answer the following questions. (Justify your answers.)

(a) Give the definition and two equivalent conditions for controllability.

A system is controllable if for finite any \( Ax_1, \ U \) an input that takes the system from 0 to \( x(t_1) = x_1 \),

A system is controllable if \([B \ AB ... A^{n-1}B]\) has full row rank, or equivalently, if \([A - \alpha I \ B]\) has full row rank for every \( \alpha \in \text{spec}(A) \).

(b) Consider the system

\[
\dot{x} = \begin{bmatrix} 3j & 0 \\ 0 & -3j \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u. \tag{11}
\]

Indicate which values of the sampling period \( T \), if any, should be avoided.

To preserve controllability (check: \( \begin{bmatrix} 1 & 3j \\ 1 & -3j \end{bmatrix} \) has full rank so the continuous system is controllable) we must have \( T \neq \frac{2\pi m}{6} \) for any positive integer \( m \).