ECE 602 Solution to Homework Assignment 1

1. For a function to be linear, it must satisfy \( y = f(0) = 0 \) in addition to \( y = ax \), so only the first graph represents a linear function. However, a change of coordinates would so that \( \tilde{y} = y - y_0 \) (using the parameter indicated in the diagram) would suffice to make the system \( \tilde{y} = ax \) linear.

2. We are given the impulse response

\[
g(t) = 2\omega \left( \frac{\sin 2\omega(t - t_0)}{2\omega(t - t_0)} \right),
\]

valid for all \( t \) for the ideal lowpass filter with constants \( \omega \) and \( t_0 \). For this system to be causal, we would need to show that the output can be represented as

\[
y(t) = \int_{-\infty}^{\infty} g(t - \tau)u(\tau)d\tau
\]

where \( g(t) = 0 \) for \( t < \tau \). It is clear that the \( g(t) \) given above is nonzero only at (an infinite number of) isolated points, hence does not satisfy the criterion for causality. Such a filter cannot be constructed for real-time use.

3. We are asked to show that the truncation operator\(^1\)

\[
y(t) = (P_\alpha u)(t) := \begin{cases} u(t) & t \leq \alpha \\ 0 & t > \alpha \end{cases}
\]

where \( \alpha \) is a fixed constant is linear and causal, but not time-invariant.

To show linearity, we note that by definition,

(a) \( P_\alpha \) maps the function \( u(t) = 0 \) \( \forall t \) to itself;
(b) \( (P_\alpha ku)(t) = k(P_\alpha u)(t) \), and
(c) \( (P_\alpha(u_1 + u_2))(t) = (P_\alpha u_1)(t) + (P_\alpha u_2)(t) \).

To show causality, we note that by its definition \( (P_\alpha u)(t) \) depends only on \( t \) itself, not on past or future values of \( t \).

To show that the operator is not time invariant, let

\[
u(t) = \begin{cases} e^{-t} & t \geq 0 \\ e^{t} & t < 0 \end{cases}
\]

Then the function \( (P_\alpha u)(t) \) does not equal the function \( (P_\beta u)(t) \) unless \( \alpha = \beta \), so the operator is not time invariant.

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\(^1\)An operator is a function that maps functions to functions.
4. We are asked to consider an initially relaxed system for which \( y = Hu \) for some operator \( H \). We are asked to show that if \( H \) is causal, then

\[
P_\alpha y = P_\alpha Hu = P_\alpha HP_\alpha u. \tag{5}
\]

We are also asked whether \( P_\alpha Hu = HP_\alpha u \).

If \( H \) is causal, then \( Hu(t) \) does not depend on any future values of \( t \), so as long as \( H0 = 0 \), \( P_\alpha Hu = P_\alpha HP_\alpha u \) because once \( t > \alpha \), \( HP_\alpha u \) is already zero and truncating it again has no further effect.

To show that, in general, \( P_\alpha Hu \neq HP_\alpha u \), consider \( H(t) = \int_{t-1}^{t} u(\tau)d\tau \). For this \( H \),

\[
HP_\alpha u(t) = \begin{cases} 
\int_{t}^{\alpha} u(\tau)d\tau & t \leq \alpha \\
\int_{\alpha}^{t} u(\tau)d\tau & t - 1 \leq \alpha \leq t \\
0 & t > \alpha
\end{cases}
\]

but

\[
P_\alpha Hu = \begin{cases} 
\int_{t}^{\alpha} u(\tau)d\tau & t \leq \alpha \\
0 & t > \alpha
\end{cases}
\]

so \( P_\alpha Hu \neq HP_\alpha u \).

5. We are given a system with input \( u \) and output \( y \). First we consider the case where \( x(0) \neq 0 \). The input is applied for \( t \geq 0 \) only, i.e. is zero for \( t < 0 \). We are told the following:

(a) The output \( y \) obtained in response to input \( u_1 + u_2 \) is the sum of the outputs \( y_1 \) and \( y_2 \) corresponding to the individual inputs \( u_1 \) and \( u_2 \).

(b) The output \( y \) obtained in response to the input \( (u_1 + u_2)/2 \) is one half the sum of the outputs \( y_1 \) and \( y_2 \) corresponding to the individual inputs \( u_1 \) and \( u_2 \).

(c) The output \( y \) obtained in response to input \( u_1 - u_2 \) is the difference of the outputs \( y_1 \) and \( y_2 \) corresponding to the individual inputs \( u_1 \) and \( u_2 \).

Which of these are true? Unfortunately, the text neglects to mention that the system is supposed to be linear. Hence we cannot say anything whatever about the truth or falsehood of any of the statements.

Now, assuming that the system is linear, if \( x(0) \) is not zero, then in the first statement, \( y_1 \) and \( y_2 \) each represent the output corresponding to \( x(0) \) and the applied input. However, applying \( u_1 + u_2 \) to the system starting in state \( x(0) \) corresponds to, for example, the sum of the output obtained for input \( u_1 \) starting at state \( x(0) \) and the output obtained for input \( u_2 \) starting at state \( x(0) = 0 \). A similar problem occurs for the third statement. Only the second is true.

If \( x(0) = 0 \), then all the statements are true.
6. We are given the system described by
\[ y(t) = \begin{cases} 
\frac{u^2(t)}{u(t-1)} & u(t-1) \neq 0 \\
0 & u(t-1) = 0 
\end{cases} \tag{8} \]
for all \( t \). We are asked to show that the system is homogeneous but not additive.

Let’s start with additivity. Let \( u_1 \) and \( u_2 \) be arbitrary. Then
\[ y_1(t) = \begin{cases} 
\frac{u_1^2(t)}{u_1(t-1)} & u_1(t-1) \neq 0 \\
0 & u_1(t-1) = 0 
\end{cases} \tag{9} \]
and
\[ y_2(t) = \begin{cases} 
\frac{u_2^2(t)}{u_2(t-1)} & u_2(t-1) \neq 0 \\
0 & u_2(t-1) = 0 
\end{cases} \tag{10} \]
On the other hand,
\[ y(t) = \begin{cases} 
\frac{(u_1+u_2)^2(t)}{(u_1+u_2)(t-1)} & (u_1 + u_2)(t - 1) \neq 0 \\
0 & (u_1 + u_2)(t - 1) = 0 
\end{cases} \tag{11} \]
is clearly not equal to the sum of \( y_1 \) and \( y_2 \) so the system is not additive.

Now let’s try homogeneity:
\[ y(t) = \begin{cases} 
\frac{u^2(t)}{u(t-1)} & u(t-1) \neq 0 \\
0 & u(t-1) = 0 
\end{cases} \tag{12} \]
and the output corresponding to \( ku \) is actually
\[ ky(t) = \begin{cases} 
\frac{(ku)^2(t)}{(ku)(t-1)} & (ku)(t - 1) \neq 0 \\
0 & (ku)(t - 1) = 0 
\end{cases} \tag{13} \]
because the \( k \) in the denominator cancels one of the \( k \)’s in the numerator and \( ku(t - 1) = 0 \) if \( u(t - 1) = 0 \).

7. We are asked to show that for rational numbers \( \alpha \) additivity implies homogeneity.

This is noted by showing first that for \( \alpha = m/n \), additivity implies that the output corresponding to input \( mu \) and initial condition \( mx_0 \) is \( m \) times the output corresponding to input \( u \) and initial condition \( x_0 \). Then if we apply input \( u = nv \) and initial condition \( x_0 = nx_{00} \) we obtain output \( n \) times the output obtained for input \( v \) and initial condition \( x_{00} \). Thus the output obtained by applying input \( u = v \) with initial condition \( x_{00} \) yields output \( 1/n \) times that for \( u = nv \) and \( x_0 = nx_{00} \). Combining these we find that the output corresponding to input \( \alpha u \) and initial condition \( \alpha x_0 \) is \( m \) times the output for \( u/n \) and \( x_0/n \) which is \( 1/n \) times that for \( u \) and \( x_0 \). Thus the output corresponding to \( \alpha u \) and \( \alpha x_0 \) is \( \alpha \) times the input for \( u \) and \( x_0 \) for any rational number \( \alpha \).
8. We are asked to let \( g(t, \tau) = g(t + \alpha, \tau + \alpha) \) for all \( t, \tau \), and \( \alpha \) and show that \( g(t, \tau) \) depends only on \( t - \tau \). If \( g(t, \tau) = g(t + \alpha, \tau + \alpha) \) for all \( \alpha \), then it must be true for \( \alpha = -\tau \), which requires \( g(t, \tau) = g(t - \tau, 0) \).

9. We are given a system whose impulse response is graphed in the \( t-g(t) \) plane as a triangle with base extending from \( t = 0 \) to \( t = 2 \) and maximum height \( b < 1 \) occurring at \( t = 1 \). We are asked to find the zero-state response of the system to an input which is \( 1 \) from \( t = 0 \) to \( t = 1 \) and \( -1 \) from \( t = 1 \) to \( t = 2 \). We simply convolve the input with the impulse response, which involves calculating the value of the function in 5 distinct regions. Obviously the output will be zero for \( t < 0 \) and for \( t > 4 \). I have used Maple to help with the integration below.

For \( 0 \leq t \leq 1 \), the output will be

\[
\int_{0}^{t} g(t - \tau)u(\tau)d\tau = \int_{0}^{t} b\tau d\tau = bt^2/2. \tag{14}
\]

For \( 1 < t \leq 2 \), the output will be

\[
\int_{0}^{t} g(t - \tau)u(\tau)d\tau = \int_{0}^{t-1} -b\tau d\tau + \int_{t-1}^{t} b(2 - \tau)d\tau \tag{15}
\]

\[= \cdots \tag{16}\]

\[= -2b + 4bt - \frac{3}{2}bt^2. \tag{17}\]

For \( 2 < t < 3 \)

\[
\int_{0}^{t} g(t - \tau)u(\tau)d\tau = \int_{t-2}^{1} -b\tau d\tau \int_{1}^{t-1} -b(2 - \tau)d\tau + \int_{t-1}^{2} -b(2 - \tau)d\tau \tag{18}
\]

\[= \cdots \tag{19}\]

\[= 10b - 8bt + \frac{3}{2}bt^2 \tag{20}\]

For \( 3 < t < 4 \)

\[
\int_{0}^{t} g(t - \tau)u(\tau)d\tau = \int_{t-2}^{2} -b(2 - \tau)d\tau \tag{21}
\]

\[= \cdots \tag{22}\]

\[= -8b + 4bt - bt^2/2. \tag{23}\]

The Maple code I used was

\[
> \text{restart:} \\
> \text{int(b*tau,tau=0..t);} \\
2 \\
bt
\]
\[ \int(-b\tau, \tau=0..(t-1)) + \int(b\tau, \tau=(t-1)..1) + \int(b(2-\tau), \tau=1..t); \]
\[ \int(-b\tau, \tau=(t-2)..1) + \int(-b(2-\tau), \tau=1..(t-1)) + \int(b(2-\tau), \tau=(t-1)..2); \]

\[
\begin{align*}
\frac{b(t-1)}{2} - & \frac{b(1-(t-1))}{2} - \frac{b(t-1)}{2} + 2 \frac{b(t-1)}{2} \\
= & -\frac{3}{2} b t + 4 b t - 2 b
\end{align*}
\]

\[
\begin{align*}
\frac{b(1-(t-2))}{2} - & \frac{b((t-1)-1)}{2} - 2 \frac{b(t-2)}{2} \\
& \frac{b(4-(t-1))}{2} + 2 \frac{b(3-t)}{2}
\end{align*}
\]

\[
\begin{align*}
\frac{b(4-(t-2))}{2} - & \frac{2 b(4-t)}{2} \\
= & 10 b + 3/2 b t - 8 b t
\end{align*}
\]

\[
\begin{align*}
\int(-b(2-\tau), \tau=(t-2)..2); \\
\frac{b(4-(t-2))}{2} - & \frac{2 b(4-t)}{2}
\end{align*}
\]
> simplify(int(-b*(2-tau),tau=(t-2)..2));

\[
\frac{2}{-1/2 \ b \ t + 4 \ b \ t - 8 \ b}
\]

10. We are asked to find the transfer function and the impulse response of the system

\[
\ddot{y} + 2\dot{y} - 3y = \dot{u} - u.  \tag{24}
\]

The transfer function is\(^\textsuperscript{2}\)

\[
\hat{g}(s) = \hat{y}(s) = \frac{s - 1}{s^2 + 2s - 3} = \frac{s - 1}{(s + 3)(s - 1)} = \frac{1}{s + 3}.  \tag{25}
\]

The impulse response is

\[
g(t) = \mathcal{L}^{-1}\left\{ \frac{1}{s + 3} \right\} = \begin{cases} e^{-3t} & t \geq 0 \\ 0 & t < 0 \end{cases}.  \tag{26}
\]

11. Show that the impulse response of a LTI system is equal to the derivative of its unit step response. Because \(u(t-\tau)\) takes the value 1 on the interval 0 to \(t\),

\[
y(t) = \int_0^t g(\tau)u(t-\tau)d\tau = \int_0^t g(\tau)d\tau  \tag{27}
\]

we have that

\[
\frac{d}{dt}y(t) = \frac{d}{dt} \int_0^t g(\tau)d\tau = g(t).  \tag{28}
\]

The last problem was to verify that the circuit shown in Figure 2.7 of the text did represent the state space equations

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
2 & -0.3 \\
1 & -8
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix}
-2 \\
0
\end{bmatrix} u  \tag{29}
\]

\[
y = \begin{bmatrix}
-2 & 3
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2
\end{bmatrix} + 5u  \tag{30}
\]

that it was alleged to represent. The verification is obtained by breaking the system into smaller components as shown on the next page.

\(^{2}\text{We assume zero initial conditions when computing the transfer function.}\)
$y = -2x_1 + 3x_2 + 5u$

$\chi_2 = x_1 - 8x_2$

$RC = 1$

Figure 2.7: OP-amp implementation of (2.17) and (2.18).