ECE 602 Solution to Assignment 2

Consider the system given by

\[ \dot{x} = f(x, u) \]  \hspace{1cm} (1)

where (1) is obtained from

\[ \ddot{\theta} = -\frac{2\dot{\theta}}{r} + \frac{2\dot{\theta}\sin \phi}{\cos \phi} + \frac{u_\theta}{mr \cos \phi} \]  \hspace{1cm} (3)

\[ \ddot{\phi} = -\dot{\theta}^2 \cos \phi \sin \phi - \frac{2\dot{\phi}}{r} + \frac{u_\phi}{mr} \]  \hspace{1cm} (4)

where \( k \) is an orbit parameter. The equations represent the dynamic behavior of a satellite.

We would like to obtain linear state equations for its motion in a stable equatorial orbit where \( r \) and \( \dot{\theta} \) are constant and \( \phi, \dot{\phi}, \) and \( u \) are zero. Linearize the system in the following steps:

1. What are the dimensions of the state vector \( x \) and the input vector \( u \)?

   We have state variables \( r, \dot{r}, \theta, \dot{\theta}, \phi, \) and \( \dot{\phi} \) so the state vector is \( 6 \times 1 \). The inputs are \( u_r, u_\theta, \) and \( u_\phi, \) so the input vector is \( 3 \times 1 \).

2. Select an appropriate state vector \( x \) and identify the components \( f_i \) of the function \( f \) that give \( \dot{x}_i = f_i(x, u) \).

   One possibility is \( x = [r, \dot{r}, \theta, \dot{\theta}, \phi, \dot{\phi}]^T \) and \( u = [u_r, u_\theta, u_\phi]^T \). Then

   \[ \dot{x}_1 = f_1(x, u) = \dot{r} = x_2 \]  \hspace{1cm} (5)

   \[ \dot{x}_2 = f_2(x, u) = r\dot{\theta}^2 \cos^2 \phi + r\dot{\phi}^2 - \frac{k}{r^2} + \frac{u_r}{m} \]  \hspace{1cm} (6)

   \[ \dot{x}_3 = f_3(x, u) = \dot{\theta} = x_4 \]  \hspace{1cm} (7)

   \[ \dot{x}_4 = f_4(x, u) = \ddot{\theta} = -\frac{2\dot{\theta}}{r} + \frac{2\dot{\theta}\sin \phi}{\cos \phi} + \frac{u_\theta}{mr \cos \phi} \]  \hspace{1cm} (8)

   \[ \dot{x}_5 = f_5(x, u) = \dot{\phi} = x_6 \]  \hspace{1cm} (9)

   \[ \dot{x}_6 = f_6(x, u) = \ddot{\phi} = -\dot{\theta}^2 \cos \phi \sin \phi - \frac{2\dot{\phi}}{r} + \frac{u_\phi}{mr} \]  \hspace{1cm} (10)
Of course, if I had chosen a different state vector, the $f_i$ would have had to change accordingly. Another common choice for the state vector would be $\mathbf{x} = [r, \theta, \phi, \dot{r}, \dot{\theta}, \dot{\phi}]^T$.

3. Determine $\partial f_i/\partial u_j$ for all $i$ and $j$.

We can determine these by inspection. We have

$$\frac{\partial f_1}{\partial u_j} = 0 \quad \forall j \in \{1, 2, 3\}$$

(11)

$$\frac{\partial f_3}{\partial u_j} = 0 \quad \forall j \in \{1, 2, 3\}$$

(12)

$$\frac{\partial f_5}{\partial u_j} = 0 \quad \forall j \in \{1, 2, 3\},$$

(13)

and,

$$\frac{\partial f_2}{\partial u_1} = \frac{1}{m}$$

(14)

$$\frac{\partial f_3}{\partial u_2} = \frac{1}{mx_1 \cos x_5}$$

(15)

$$\frac{\partial f_5}{\partial u_3} = \frac{1}{mx_1}.$$  

(16)

Finally, all remaining partials are zero.

4. Determine $\partial f_i/\partial x_j$ for all $i$ and $j$.

I recommend learning to use a symbolic mathematics software package like Maple in order to computing partials like this. It is easy to make a mistake by hand. (It is also easy to make a mistake in Maple, but if you do the calculations both ways and they don’t match, you then have a hint as to where the error lies.)

Here are my hand-computed partials (I’ve listed only the nonzero ones):

$$\frac{\partial f_1}{\partial x_2} = 1$$

(17)

$$\frac{\partial f_2}{\partial x_1} = x_4^2 \cos^2 x_5 + x_6^2 + 2 \frac{k}{x_4^4}$$

(18)

$$\frac{\partial f_2}{\partial x_4} = 2x_1 x_4 \cos^2 x_5$$

(19)

$$\frac{\partial f_2}{\partial x_5} = 2x_1 x_4^2 \cos x_5 \sin x_5$$

(20)

$$\frac{\partial f_2}{\partial x_6} = 2x_1 x_6$$

(21)

$$\frac{\partial f_3}{\partial x_4} = 1$$

(22)

$$\frac{\partial f_4}{\partial x_1} = \frac{2x_2 x_4}{x_1^2} - \frac{u_2}{mx_1^2 \cos x_5}$$

(23)
\[ \frac{\partial f_4}{\partial x_2} = -\frac{2x_4}{x_1} \quad (24) \]
\[ \frac{\partial f_4}{\partial x_4} = -\frac{2x_2}{x_1} + \frac{2x_6 \sin x_5}{\cos x_5} \quad (25) \]
\[ \frac{\partial f_4}{\partial x_5} = 2x_4 - \frac{2x_4x_6 \sin^2 x_5}{\cos^2 x_5} - \frac{u_2 \sin x_5}{mx_1 \cos^2 x_5} \quad (26) \]
\[ \frac{\partial f_4}{\partial x_6} = \frac{2x_4 \sin x_5}{\cos x_5} \quad (27) \]
\[ \frac{\partial f_5}{\partial x_6} = 1 \quad (28) \]
\[ \frac{\partial f_6}{\partial x_1} = \frac{2x_2x_6}{x_1^2} - \frac{u_3}{mx_1^2} \quad (29) \]
\[ \frac{\partial f_6}{\partial x_2} = -\frac{2x_6}{x_1} \quad (30) \]
\[ \frac{\partial f_6}{\partial x_4} = -2x_4 \cos x_5 \sin x_5 \quad (31) \]
\[ \frac{\partial f_6}{\partial x_5} = -x_2^2 (\cos^2 x_5 - \sin^2 x_5) \quad (32) \]
\[ \frac{\partial f_6}{\partial x_6} = -\frac{2x_2}{x_1} \quad (33) \]

5. Now consider the equilibrium state \( \mathbf{x}_e \) with

\[ r_e = r_0, \quad (34) \]
\[ \theta_e = \omega_0 t, \quad (35) \]
\[ \dot{\theta}_e = \omega_0, \quad \text{and} \]
\[ \dot{r}_e = \phi_e = \dot{\phi}_e = 0 \quad (37) \]

and \( k = \omega_0^2 r_0^3 \). From above, \( \mathbf{u}_e = 0 \).

(a) Determine \( \frac{\partial f_i}{\partial u_j} \bigg|_{(\mathbf{x}, \mathbf{u})=(\mathbf{x}_e, \mathbf{u}_e)} \) for all \( i \) and \( j \) and the matrix \( \mathbf{B} \).

We can determine these by inspection. We had

\[ \frac{\partial f_1}{\partial u_j} = 0 \quad \forall j \in \{1, 2, 3\} \quad (38) \]
\[ \frac{\partial f_3}{\partial u_j} = 0 \quad \forall j \in \{1, 2, 3\} \quad (39) \]
\[ \frac{\partial f_5}{\partial u_j} = 0 \quad \forall j \in \{1, 2, 3\}, \quad (40) \]

so the corresponding elements of \( \mathbf{B} \) will be zero. Evaluating at \( (\mathbf{x}_e, \mathbf{u}_e) \), we obtain

\[ \frac{\partial f_2}{\partial u_1} \bigg|_{(\mathbf{x}_e, \mathbf{u}_e)} = \frac{1}{m} \quad (41) \]
\[ \frac{\partial f_3}{\partial u_2}(x_e, u_e) = \frac{1}{mr_0} \]  
\[ \frac{\partial f_5}{\partial u_3}(x_e, u_e) = \frac{1}{mr_0} , \]  
and all remaining partials are zero. Thus

\[
B = \begin{bmatrix}
0 & 0 & 0 \\
\frac{1}{m} & 0 & 0 \\
0 & 0 & 0 \\
0 & \frac{1}{mr_0} & 0 \\
0 & 0 & 0 \\
0 & 0 & \frac{1}{mr_0}
\end{bmatrix}.
\]

(b) Determine \( \frac{\partial f_i}{\partial x_j} \) for all \( i \) and \( j \) and the matrix \( A \).

Some of the partials that don’t depend on the equilibrium:

\[
\frac{\partial f_1}{\partial x_2}(x_e, u_e) = 1, \]
\[
\frac{\partial f_3}{\partial x_4}(x_e, u_e) = 1, \text{ and}
\]
\[
\frac{\partial f_5}{\partial x_6}(x_e, u_e) = 1 .
\]

Many of the partials become zero when we substitute

\[
x_{2,e} = x_{5,e} = x_{6,e} = u_{1,e} = u_{2,e} = u_{3,e} = 0,
\]

specifically,

\[
\frac{\partial f_2}{\partial x_3}(x_e, u_e) = 2r_0 \omega_0^2 \cos 0 \sin 0 = 0,
\]
\[
\frac{\partial f_4}{\partial x_1}(x_e, u_e) = \frac{2(0) \omega_0}{r_0^2} - \frac{0}{mr_0^2 \cos 0} = 0,
\]
\[
\frac{\partial f_4}{\partial x_4}(x_e, u_e) = -\frac{2(0)}{r_0} + \frac{2(0) \sin 0}{\cos 0} = 0,
\]
\[
\frac{\partial f_4}{\partial x_5}(x_e, u_e) = 2 \omega_0 - \frac{2 \omega_0 \sin^2 0}{\cos^2 0} - \frac{0 \sin 0}{mr_0 \cos^2 0} = 0,
\]
\[
\frac{\partial f_4}{\partial x_6}(x_e, u_e) = \frac{2 \omega_0 \sin 0}{\cos 0} = 0,
\]
\[
\frac{\partial f_6}{\partial x_1}(x_e, u_e) = \frac{2(0) \omega_0}{r_0^2} - \frac{0}{mr_0^2} = 0,
\]
\[
\frac{\partial f_6}{\partial x_6}(x_e, u_e) = -\frac{2(0)}{r_0} = 0,
\]
\[
\frac{\partial f_6}{\partial x_4}(x_e, u_e) = -2 \omega_0 \cos 0 \sin 0 = 0, \text{ and}
\]
\[
\frac{\partial f_6}{\partial x_6}(x_e, u_e) = -\frac{2(0)}{r_0} = 0 .
\]
Finally, the partials that are nonzero at the equilibrium are:

\[
\partial f_2/\partial x_1\bigg|_{x_\epsilon, u_\epsilon} = \omega_0^2 \cos^2 0 + 0^2 + 2 \frac{\omega_0^2 r_0^3}{r_0^3} = \omega_0^2 + 2 \omega_0^2, \quad (57)
\]

\[
\partial f_2/\partial x_4\bigg|_{x_\epsilon, u_\epsilon} = 2r_0 \omega_0 \cos^2 0 = 2r_0 \omega_0, \quad (58)
\]

\[
\partial f_4/\partial x_2\bigg|_{x_\epsilon, u_\epsilon} = -\frac{2 \omega_0}{r_0} = -2 \frac{\omega_0}{r_0}, \quad \text{and} \quad (59)
\]

\[
\partial f_6/\partial x_5\bigg|_{x_\epsilon, u_\epsilon} = -\omega_0^2 (\cos^2 0 - \sin^2 0) = -\omega_0^2. \quad (60)
\]

Thus the A matrix is given by

\[
A = \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
3 \omega_0^2 & 0 & 0 & 2r_0 \omega_0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & -2 \omega_0/r_0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & -\omega_0^2 & 0 \\
\end{bmatrix}. \quad (61)
\]

6. Write the linearized state equations in matrix form.

The linearized state equations describe deviations of the states from the nominal (equilibrium) values so we define

\[
\ddot{\mathbf{r}} = \dot{\mathbf{r}} - r_0, \quad \text{then} \quad (62)
\]

\[
\ddot{\mathbf{r}} = \dot{\mathbf{r}} - \dot{r}_0 = \dot{\mathbf{r}}, \quad \text{and} \quad (63)
\]

\[
\ddot{\theta} = \dot{\theta} - \omega_0 t, \quad \text{then} \quad (64)
\]

\[
\ddot{\theta} = \dot{\theta} - \dot{\theta}_0 = \dot{\theta}, \quad (65)
\]

and, of course, since \( \phi_e \) and \( \dot{\phi}_e \) are zero, \( \ddot{\phi} = \phi \) and \( \ddot{\phi} = \dot{\phi} \). Similarly, let’s define \( \ddot{\mathbf{u}} = \mathbf{u} \) since \( \mathbf{u}_e = \mathbf{0} \). Then, finally, we have the linearized state equation

\[
\dot{\mathbf{x}} = A\mathbf{x} + B\ddot{\mathbf{u}}. \quad (66)
\]