ECE 602 Lecture Notes:
Cayley-Hamilton Examples

The Cayley Hamilton Theorem states that a square $n \times n$ matrix $A$ satisfies its own characteristic equation. Thus, we can express $A^n$ in terms of a finite set of lower powers of $A$. This fact leads to a simple way of calculating the value of a function evaluated at the matrix. This method is given in Theorem 3.5 of the textbook\(^1\). Here we give a couple of examples.

We will consider polynomial functions $f(A)$ and the exponential function $e^{At}$. We’ll work with the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which has the convenient property that

$$A^k = \begin{bmatrix} 1 & k & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$ (2)

The characteristic equation of $A$ can be obtained easily by noting the second and third rows each have only a single nonzero entry. Using the second we have

$$\Delta(\lambda) = (s - 1)^3,$$ (3)

so the matrix has a single eigenvalue 1 with multiplicity three.

Because the matrix is $3 \times 3$ we will use the polynomial

$$h(\lambda) := \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$$ (4)

in our calculations. We select the notation $f^{(k)}(\lambda) := d^k f / d\lambda^k$.

**Example 1** Consider the polynomial $f(\lambda) = \lambda^5 - 1$.

We obtain three linear equations in three unknowns as follows. Because the eigenvalue of $A$ has multiplicity three we must use two derivatives of $f$ and $h$. Our equations are

$$f(\lambda) = \lambda^5 - 1 \quad h(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$$ (5)

$$f^{(1)}(\lambda) = 5\lambda^4 \quad h^{(1)}(\lambda) = \beta_1 + 2 \beta_2 \lambda$$ (6)

$$f^{(2)}(\lambda) = 20\lambda^3 \quad h^{(2)}(\lambda) = 2 \beta_2.$$ (7)

We obtain
\[ f(1) = 0 = \beta_0 + \beta_1 + \beta_2 = h(1) \] (8)
\[ f^{(1)}(1) = 5 = \beta_1 + 2\beta_2 = h^{(1)}(1) \] (9)
\[ f^{(2)}(1) = 20 = 2\beta_2 = h^{(2)}(1) \] (10)

Solving for the \( \beta \)s we find
\[ \beta_2 = 10, \quad \beta_1 = -15, \quad \beta_0 = 5. \] (11)

Thus
\[ f(A) = \beta_0 A^0 + \beta_1 A^1 + \beta_2 A^2 \] (12)
\[ = 5I - 15A + 10A^2 \] (13)
\[ = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \] (14)
\[ = A^5 - I. \] (15)

**Example 2** Consider the function \( f(\lambda) = e^{\lambda t} \).

We obtain three linear equations in three unknowns as follows. Because the eigenvalue of \( A \) has multiplicity three we must use two derivatives of \( f \) and \( h \). Remembering that we are taking derivatives with respect to \( \lambda \), our equations are
\[ f(\lambda) = e^{\lambda t} \quad h(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2 \] (16)
\[ f^{(1)}(\lambda) = e^{\lambda t} \quad h^{(1)}(\lambda) = \beta_1 + 2\beta_2 \lambda \] (17)
\[ f^{(2)}(\lambda) = t^2 e^{\lambda t} \quad h^{(2)}(\lambda) = 2\beta_2. \] (18)

We obtain
\[ f(1) = e^t = \beta_0 + \beta_1 + \beta_2 = h(1) \] (19)
\[ f^{(1)}(1) = te^t = \beta_1 + 2\beta_2 = h^{(1)}(1) \] (20)
\[ f^{(2)}(1) = t^2 e^t/2 = 2\beta_2 = h^{(2)}(1). \] (21)

Solving for the \( \beta \)s we find
\[ \beta_2 = t^2 e^t/2, \quad \beta_1 = te^t - t^2 e^t, \quad \beta_0 = e^t - te^t + t^2 e^t/2. \] (22)

Thus
\[ f(A) = \beta_0 A^0 + \beta_1 A^1 + \beta_2 A^2 \] (23)
\[ = e^t (1 - t + t^2/2) I + e^t (t - t^2) A + e^t (t^2/2) A^2 \] (24)
\[ = \begin{bmatrix} e^t & te^t & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^t \end{bmatrix}. \] (25)
Example 3 Computation of $(sI - A)^{-1}$ for

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

using equivalence on the spectrum of $A$.

First let’s find the spectrum of $A$.

$$|sI - A| = \begin{bmatrix} s & 1 \\ -1 & s + 2 \end{bmatrix} = s(s + 2) - (-1) = s^2 + 2s + 1 = (s + 1)^2$$

so $A$ has eigenvalue $\lambda = -1$ with multiplicity 2.

Next, let $f(\lambda) = (s - \lambda)^{-1}$ and, as always, $h(\lambda) = \beta_0 + \beta_1 \lambda$.

Then $f^{(1)}(\lambda) = -(-1)(s - \lambda)^{-2}$ and $h^{(1)}(\lambda) = \beta_1$.

Substituting the eigenvalue for $\lambda$ yields

$$\beta_1 = (s + 1)^{-2}$$
$$\beta_0 = (s + 1)^{-1} - (-1)(s + 1)^{-2} = (s + 2)(s + 1)^{-2}$$

Thus $h(\lambda) = (s + 2)(s + 1)^{-2} + (s + 1)^{-2} \lambda$ and substituting $A$ for $\lambda$ yields

$$f(A) = (sI - A)^{-1}$$
$$= (s + 2)(s + 1)^{-2}I + (s + 1)^{-2}A$$

$$= \begin{bmatrix} \frac{s + 2}{(s + 1)^2} & \frac{-1}{(s + 1)^2} \\ \frac{1}{(s + 1)^2} & \frac{s + 2 - 2}{(s + 1)^2} \end{bmatrix}.$$