Using the Lagrangian to obtain Equations of Motion

In Section 1.5 of the textbook, Zak introduces the Lagrangian $L = K - U$, which is the difference between the kinetic and potential energy of the system. He then proceeds to obtain the Lagrange equations of motion in Cartesian coordinates for a point mass subject to conservative forces, namely,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \quad i = 1, 2, 3. \tag{1}$$

(Any nonconservative forces acting on the point mass would show up on the right hand side.)

Here's how the text gets from the definition to the result.

We know that for a point mass, force is equal to mass times acceleration,

$$\mathbf{F} = m\mathbf{a} = m\ddot{\mathbf{x}} = m\frac{d\dot{\mathbf{x}}}{dt},$$

and work is equal to the integral over distance of the applied force. We can substitute for the force to obtain

$$\begin{aligned} W &= \int_A^B \mathbf{F} \cdot \mathbf{dx} & (2) \\ &= \int_A^B m\ddot{\mathbf{x}} \cdot \mathbf{dx} & (3) \\ &= \int_A^B m\dot{\mathbf{x}} \cdot \mathbf{d\dot{x}} & (4) \end{aligned}$$

where we have played fast and loose with the derivatives to conclude that

$$\ddot{x}dx = (d\dot{x}/dt)dx = d\dot{x}(dx/dt) = \dot{x}d\dot{x}.$$

Assuming conservative forces, we can then integrate to obtain

$$W = \frac{m}{2}\left(\dot{\mathbf{x}}^T\dot{\mathbf{x}}\right)\Big|_A^B \tag{5}$$

so the work is the difference between the kinetic energy at point B and that at point A.

Conservation of energy requires that an increase in kinetic energy must be balanced by decrease in potential energy so we can write

$$-\int_A^B \mathbf{F} \cdot \mathbf{dx} = \Delta U \tag{6}$$

and thus

$$\mathbf{F} = -\nabla U,$$

where we use the notation

$$\nabla U := \left[\frac{\partial U}{\partial x_1} \ \frac{\partial U}{\partial x_2} \ \frac{\partial U}{\partial x_3} \right]^T.$$

We have used the fact that if we measure change in potential energy with respect to a constant reference, the derivative of the constant reference is zero so we have

$$\nabla(\Delta U) = \nabla U.$$

We're almost ready to rewrite Newton's equation in its Lagrangian form.

We know that

$$\frac{\partial K}{\partial \dot{x}_i} = m\dot{x}_i \tag{7}$$

so Newton's law becomes

$$F = -(\nabla U)_i = \frac{d}{dt}\left(\frac{\partial K}{\partial \dot{x}_i}\right) = m\ddot{x}_i \quad i = 1, 2, 3. \tag{8}$$

Now with $L = K - U$, we see that K does not depend on position and U does not depend on velocity, so

$$\frac{\partial L}{\partial \dot{x}_i} = \frac{\partial K}{\partial \dot{x}_i} \tag{9}$$

$$\frac{\partial L}{\partial x_i} = -\frac{\partial U}{\partial x_i} \tag{10}$$

so Newton's equation can be rewritten as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \quad i = 1, 2, 3 \tag{11}$$

as asserted earlier.

Next, in Section 1.6, Zak extends the above analysis to generalized coordinates by expressing each of the x_i in terms of new coordinates q_i. By the chain rule we then have

$$\dot{x}_i = \frac{\partial x_i}{\partial q_1}\dot{q}_1 + \frac{\partial x_i}{\partial q_2}\dot{q}_2 + \frac{\partial x_i}{\partial q_3}\dot{q}_3 \tag{12}$$

and after repeating the derivation with the new coordinates q_i we obtain, (surprise, surprise,)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0 \quad i = 1, 2, 3. \tag{13}$$

If the applied force has a nonconservative component, the right-hand side is equal to the nonconservative component rather than zero.

Let's do a couple of simple examples to demonstrate that this is a viable method for obtaining the equations of motion.

Example 1: Pendulum

Consider a pendulum of mass m and length ℓ with angular displacement θ from the vertical. From the geometry, the expressions for the kinetic and potential energies are

$$K = \frac{1}{2}m\left(l\dot{\theta}\right)^2 \tag{14}$$
$$U = mgl\,(1-\cos\theta)\,. \tag{15}$$

Accordingly,

$$L = K - U = \frac{1}{2}m\ell^2\dot{\theta}^2 - mgl(1-\cos\theta). \tag{16}$$

The

$$\frac{\partial L}{\partial \theta} = -mg\ell\sin\theta \tag{17}$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell^2\dot{\theta} \tag{18}$$

so

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = m\ell^2\ddot{\theta} \tag{19}$$

and finally solving for θ we have

$$\ddot{\theta} = -\frac{g\sin\theta}{\ell}. \tag{20}$$

Example 2: Pendulum on Cart

This may have seemed like a very difficult way to get the equation of motion of a pendulum, so let's try a more complicated example. We hang the pendulum from a cart of mass M and position x, acted upon by a force u in the direction of x, and moving on frictionless rails.

The the x position of the pendulum is $x+\ell\sin\theta$ and the y position is $\ell\cos\theta$, so the kinetic energy is

$$K = \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left(\frac{d}{dt}\left(x+\ell\sin\theta\right)\right)^2 + \frac{1}{2}m\left(\frac{d}{dt}\left(\ell\cos\theta\right)\right)^2. \tag{21}$$

First taking the time-derivatives, then squaring, then noting that $\cos^2\theta + \sin^2\theta = 1$ we obtain

$$K = \frac{1}{2}\left(M+m\right)\dot{x}^2 + m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2. \tag{22}$$

The potential energy is as before, so

$$L = K - U = \frac{1}{2}\left(M+m\right)\dot{x}^2 + m\ell\dot{x}\dot{\theta}\cos\theta + \frac{1}{2}m\ell^2\dot{\theta}^2 - mg\ell\left(1-\cos\theta\right). \tag{23}$$

Clearly $\partial L/\partial x = 0$ and

$$\frac{\partial L}{\partial \dot{x}} = (M+m)\,\dot{x} + m\ell\dot{\theta}\cos\theta \tag{24}$$

so

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = (M+m)\,\ddot{x} + m\ell\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right) = u \tag{25}$$

Next we consider the θ direction and velocity, taking

$$\frac{\partial L}{\partial \theta} = -m\ell\dot{x}\dot{\theta}\sin\theta + mg\ell\dot{\theta}\sin\theta \tag{26}$$

and

$$\frac{\partial L}{\partial \dot{\theta}} = m\ell\dot{x}\cos\theta + m\ell^2\dot{\theta}. \tag{27}$$

Taking the time derivative yields

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \left(m\ell\ddot{x}\cos\theta - m\ell\dot{x}\dot{\theta}\sin\theta\right) + m\ell^2\ddot{\theta}. \tag{28}$$

The Lagrangian equation of motion is thus

$$m\ell\left(\ddot{x}\cos\theta + \ell\ddot{\theta} - g\sin\theta\right) = 0. \tag{29}$$

We can write this as a matrix differential equation

$$\begin{bmatrix} M+m & m\ell\cos\theta \\ \cos\theta & \ell \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} m\ell\dot{\theta}^2\sin\theta + u \\ g\sin\theta \end{bmatrix}. \tag{30}$$

Of course the cart pendulum is really a fourth order system so we'll want to define a new state vector $\begin{bmatrix} x & \dot{x} & \theta & \dot{\theta} \end{bmatrix}^T$ in order to solve the nonlinear state equation.

$$\tag{31}$$

For comparison, it will be instructive to read Section 1.7 in which Zak presents an example of a cart with inverted pendulum. Instead of using the Lagrangian equations of motion, he applies Newton's law in its usual form. There are a couple of differences between the examples. Specifically, in the example in Section 1.7

1. the pendulum is a distributed rather than point mass, and
2. frictional force on the cart wheels is considered.