

Some Notes on Generalized Eigenvectors

In lecture we discussed generalized eigenvalues and how to find them. At the top of page 60 the text¹ states that the representation of the matrix A with respect to the basis v_1 , v_2 , v_3 , and v_4 is

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}. \quad (1)$$

This may be somewhat confusing since no A matrix nor values for v_1 , v_2 , v_3 , and v_4 were specified. The reason for this is as follows. A here is any 4×4 matrix with the eigenstructure of J . It could be equal to J or it could be, for example,

$$A = \begin{bmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 1 & 0 & \lambda \end{bmatrix}, \quad (2)$$

which I obtained by applying a permutation matrix to J . Let's start with $A = J$ and find a set of generalized eigenvectors for $A = J$. The one eigenvector corresponding to the eigenvalue λ is $v_1 = [1 \ 0 \ 0 \ 0]^T$. Our next generalized eigenvector (remember that eigenvectors are considered generalized eigenvectors) should satisfy

$$(A - \lambda I)v_2 = v_1. \quad (3)$$

Letting $v_2 = [a \ b \ c \ d]^T$ we have

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (4)$$

so a is arbitrary, let's call it p , b is one, and c and d are zero. Repeating the process we have

$$(A - \lambda I)v_3 = v_2. \quad (5)$$

Letting $v_3 = [a \ b \ c \ d]^T$ we have

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad (6)$$

so a is arbitrary again, let's call it q , b is p , c is one, and d is zero.

¹Chen, C.T. *Linear System Theory and Design*, 3rd ed., Oxford University Press, 1999.

Finally, we solve

$$(A - \lambda I)v_4 = v_3. \quad (7)$$

Letting $v_4 = [a \ b \ c \ d]^T$ we have

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} q \\ p \\ 1 \\ 0 \end{bmatrix} \quad (8)$$

so a is arbitrary again, let's call it r , b is q , c is p , and d is one.

This is why the actual vectors v_1 , v_2 , v_3 , and v_4 weren't specified. We can use the vectors we found as a basis and let

$$Q = \begin{bmatrix} 1 & p & q & r \\ 0 & 1 & p & q \\ 0 & 0 & 1 & p \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (9)$$

Maple tells us that

$$Q^{-1} = \begin{bmatrix} 1 & -p & p^2 - q & -p^3 + 2pq - r \\ 0 & 1 & -p & p^2 - q \\ 0 & 0 & 1 & -p \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (10)$$

and verifies that $Q^{-1}JQ = J$, regardless of the values p , q , and r .

Suppose we started with

$$A = \begin{bmatrix} \lambda & 0 & 1 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 1 & 0 & \lambda \end{bmatrix}, \quad (11)$$

instead. Then with help from Matlab or Maple, we verify that the characteristic equation is still $p(s) = (s - \lambda)^4$ so we have the same eigenvalues. Then

$$(A - \lambda I) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (12)$$

implies that c is one, d is zero, and b is zero so again we let $a = p$. Then

$$(A - \lambda I) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} p \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (13)$$

so $c = p$, $d = 1$, and $b = 0$, so we let $a = q$. Then

$$(A - \lambda I) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} q \\ 0 \\ p \\ 1 \end{bmatrix} \quad (14)$$

so $c = q$, $d = p$, $b = 1$ and we let $a = r$. Now

$$Q = \begin{bmatrix} 1 & p & q & r \\ 0 & 0 & 0 & 1 \\ 0 & 1 & p & q \\ 0 & 0 & 1 & p \end{bmatrix} \quad (15)$$

and

$$Q^{-1} = \begin{bmatrix} 1 & -r + 2qp - p^3 & -p & p^2 - q \\ 0 & p^2 - q & 1 & -p \\ 0 & -p & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}. \quad (16)$$

Finally, Matlab verifies that $Q^{-1}AQ = J$.