Some Notes on Projections

In Chapter 0 of the text, the author introduces three different types of projections.

$\mathscr{X}$, $\mathscr{R}$, and $\mathscr{S}$ are linear spaces such that $\mathscr{X} = \mathscr{R} \oplus \mathscr{S}$, i.e. the linear space $\mathscr{X}$ is the internal direct sum of (independent) subspaces $\mathscr{R}$ and $\mathscr{S}$. This means that each $x \in \mathscr{X}$ can be expressed uniquely as $x = r + s$ where $r \in \mathscr{R}$ and $s \in \mathscr{S}$.

projection The map $x \mapsto r$ is called the projection on $\mathscr{R}$ along $\mathscr{S}$.

The resulting function is a linear map $Q : \mathscr{X} \to \mathscr{X}$ whose image is $\mathscr{R}$ and whose kernel is $\mathscr{S}$. Knowing that the image and kernel of a linear map are subspaces, one can write $\mathscr{X} = \operatorname{Im} Q \oplus \ker Q$.

natural projection The natural projection is the map $x \mapsto r$, whose codomain is the subspace $\mathscr{R}$, rather than all of $\mathscr{X}$. The natural projection $\tilde{Q} : \mathscr{X} \to \mathscr{R}$, can thus be expressed as $\tilde{Q} = \mathscr{R}|Q$.

canonical projection The canonical projection of $\mathscr{X}$ onto its factor space $\mathscr{X}/\mathscr{S}$ is the map $x \mapsto x + \mathscr{S}$, where $x + \mathscr{S} := \{y : y \in \mathscr{X},\ y - x \in \mathscr{S}\}$. The resulting function $P : \mathscr{X} \to \mathscr{X}/\mathscr{S}$ has kernel $\mathscr{S}$. The symbol $\bar{x}$ will often be used for $x + \mathscr{S}$ if the factor space is understood.

Example

Let $\mathscr{X}$ the usual Euclidean three-space, $\mathscr{R}$ the xy-plane, and $\mathscr{Z}$ the z-axis. We then have $\mathscr{X} = \mathscr{R} \oplus \mathscr{S}$. Accordingly we can express a point $(x, y, z) \in \mathscr{X}$, as $(x, y) + z$.

$Qx = (x, y, 0)$ is an element of the three-dimensional space whose z coordinate is zero.

$\tilde{Q}x = (x, y)$ is an element of the xy-plane.

$Px = (x, y) + \mathscr{S}$ is a vertical line through the point $(x, y, 0)$.