# Proposition 6.1 of Wonham's Linear Multivariable Control

Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ \dot{y}(t) &= Cx(t), \\ \dot{z}(t) &= Dx(t), \end{aligned}$$

for $t \geq 0$.

Let $\mathscr{N}$ be an $A$-invariant subspace of the state space $\mathscr{X}$ and

$$\mathfrak{B} := \{\mathscr{V} : \mathscr{V} \in \mathfrak{J}(A, B; \operatorname{Ker} D),\ A(\mathscr{V} \cap \mathscr{N}) \subset \mathscr{V}\},$$

where

$$\mathfrak{J}(A, B; \operatorname{Ker} D) := \{\mathscr{V} : \mathscr{V} \in \mathfrak{J}(A, B; \mathscr{X}),\ \mathscr{V} \in \operatorname{Ker} D\},$$

and $\mathfrak{J}(A, B; \mathscr{X})$ is the class of $(A, B)$-invariant subspaces of the state space.

**Proposition 6.1** Let

$$\mathscr{V}_0 := \cap_{i=1}^{n} A^{-i+1} (\mathscr{N} \cap \operatorname{Ker} D).$$

Then each subspace $\mathscr{V}^M$ is of the form

$$\mathscr{V}^M = \mathscr{V}_0 \oplus \mathscr{V}_1,$$

where

$$\mathscr{V}_1 = \sup \left\{\mathscr{V} : \mathscr{V} \subset \mathscr{W} \cap A^{-1} (\mathscr{B} + \mathscr{V}_0 + \mathscr{V})\right\}$$

and $\mathscr{W}$ is a suitable complement of $\mathscr{N} \cap \operatorname{Ker} D$ in the kernel of $D$.

**Proof**

Let $\mathscr{V}^M$ be a maximal element of $\mathfrak{B}$ and $\mathscr{V}_1^M$ and $\mathscr{W}_1$ be such that

$$\mathscr{V}^M = \mathscr{V}^M \cap \mathscr{N} \oplus \mathscr{V}_1^M \tag{1}$$

$$\operatorname{Ker} D = \operatorname{Ker} D \cap \mathscr{N} \oplus \mathscr{V}_1^M \oplus \mathscr{W}_1 \tag{2}$$

and let $\mathscr{W} = \mathscr{V}_1^M \oplus \mathscr{W}_1$. By assumption, $\mathscr{V}_0 \subset \mathscr{N} \cap \operatorname{Ker} D$ (with $i = 1$), and is $A$-invariant. As a result of the Cayley-Hamilton theorem,

$$\mathscr{V}_0 = \sup \{\mathscr{V} : \mathscr{V} \subset \mathscr{N} \cap \operatorname{Ker} D, A\mathscr{V} \subset \mathscr{V}\}.$$

<u>Claim 1</u> $\mathscr{V}_0 = \mathscr{V}^M \cap \mathscr{N}$.

<u>Proof of Claim 1:</u> As $\mathscr{V}_0 \cap \mathscr{N}$ and is the maximal element of $\mathfrak{B}$, we can conclude that the subspace

$$\left(\mathscr{V}^M + \mathscr{V}_0\right) \cap \mathscr{N} = \mathscr{V}^M \cap \mathscr{N} + \mathscr{V}_0$$

is $A$-invariant, and hence that $\mathscr{V}^M + \mathscr{V}_0 \in \mathfrak{B}$. But $\mathscr{V}^M$ being maximal, it contains $\mathscr{V}_0$ and, $\mathscr{V}_0$ being contained in $\mathscr{N}$ we have that

$$\mathscr{V}^M \cap \mathscr{N} \supset \mathscr{V}_0.$$

Also, since $\mathscr{V}^M \cap \mathscr{N}$ is an element of the set whose maximal element is $\mathscr{V}_0$, $\mathscr{V}^M \cap \mathscr{N} \subset \mathscr{V}_0$, so $\mathscr{V}^M \cap \mathscr{N} = \mathscr{V}_0$. □

Claim 2 $\mathscr{V}_1 = \mathscr{V}_1^M$.

Proof of Claim 2: By definition, $\mathscr{V}_1^M \subset \mathscr{V}^M$, and $\mathscr{V}^M$ is $(A, B)$-invariant so

$$A\mathscr{V}_1^M \subset A\mathscr{V}^M \subset \mathscr{V} + \mathscr{B}.$$

Also by definition of $\mathscr{V}_1^M$,

$$\mathscr{V} + \mathscr{B} = \mathscr{V}^M \cap \mathscr{N} + \mathscr{V}_1^M + \mathscr{B}.$$

Then using the result of Claim 1,

$$\mathscr{V} + \mathscr{B} = \mathscr{V}_0 + \mathscr{V}_1^M + \mathscr{B},$$

so $\mathscr{V}_1^M \subset \mathscr{V}_1$.

Also by definition of $\mathscr{V}_1^M$, $\mathscr{V}_1 \subset \mathscr{W}$ and as $\mathscr{W} \cap \mathscr{N} = 0$, $\mathscr{V}_1 \cap \mathscr{N} = 0$. Then

$$(\mathscr{V}_0 \oplus \mathscr{V}_1) \cap \mathscr{N} = \mathscr{V}_0,$$

which is $A$-invariant and contained in the kernel of $D$ so

$$A\,(\mathscr{V}_0 \oplus \mathscr{V}_1) \subset \mathscr{V}_0 + \mathscr{V}_1 + \mathscr{B}.$$

Thus $\mathscr{V}_0 \oplus \mathscr{V}_1 \in \mathfrak{B}$ and since $\mathscr{V}_1^M \subset \mathscr{V}_1$ we have that $\mathscr{V}_0 \oplus \mathscr{V}_1 \supset \mathscr{V}_0 \oplus \mathscr{V}_1^M$. The quantity on the right hand side of this containment is just $\mathscr{V}^M$, which is maximal, so $\mathscr{V}_1 = \mathscr{V}_1^M$. □

This result then proves the proposition. □