# Rational Canonical Structure Example

Let's start with the simplest possible example, namely $\mathscr{X} = \mathbb{R}$, $A = 1$. Here $\sigma(A) = \{1\}$,

$$\pi(\lambda) = \alpha(\lambda) = \lambda - 1, \tag{1}$$

and $A$ is cyclic with generator 1. Let $e_1 = 1$ The number of elements in the direct product is

$$k = \max\{\dim(\mathrm{Ker}A - \lambda I)\} = 1 \tag{2}$$

so $\mathscr{X} = \mathscr{X}_1$, $A \mid \mathscr{X}_1 = A$ is cyclic, etc. as promised by Theorem 0.1.

Next, let's consider the two by two identity matrix $A = I$.

$$\mathrm{Ker}(A - I) = \mathrm{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\} \tag{3}$$

so $k = 2$. We can decompose $\mathscr{X}$ into components $\mathscr{X}_1 = \mathscr{X}_2 = \mathbb{R}$. $A|\mathscr{X}_i$, being 1 for $i \in \{1, 2\}$ is cyclic, with $\alpha_i(\lambda) = \lambda - 1$.

Next, consider

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

$$\{g\,,\, Ag\} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \tag{4}$$

forms a basis of $\mathscr{X}$. We define

$$\begin{aligned} \alpha^{(0)}(\lambda) &:= \alpha(\lambda) = \lambda^2 - (-1 + 2\lambda) && (5)\\ \alpha^{(1)}(\lambda) &:= \lambda - 2 && (6)\\ \alpha^{(2)}(\lambda) &:= 1 && (7) \end{aligned}$$

Then

$$\begin{aligned} e_1 &:= \alpha^{(1)}(A)g = (A - 2I)g && (8)\\ &= \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} && (9)\\ & \begin{bmatrix} 1 \\ -1 \end{bmatrix} && (10)\\ e_2 &:= \alpha^{(2)}(A)g = g = \begin{bmatrix} 0 \\ 1 \end{bmatrix} && (11) \end{aligned}$$

Taking $P = [e_1\; e_2]$ and noting that $P^{-1} = P$ we find that with respect to this basis,

$$\mathrm{Mat}(A) = P^{-1}AP = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$$

which is in companion form as required.

$$\text{Ker}\,(A - I) = \text{Ker}\left\{\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$$

so $k = 1$. We have found that $\mathscr{X}_1 = \mathscr{X} = \mathbb{R}^2$ and $A|\mathscr{X}_1 = A$ which is cyclic with generator $g$.