Gerlando's offers 2 kinds of crust, Tuscan (T) and Neapolitan (N). Also, each pizza may have mushrooms (M) or onions (O) as indicated in the Venn diagram.

(a) Are N and M mutually exclusive?
No. \( N = T^c \) and both \( T \) and \( T^c \) intersect \( M \).

(b) Are \( N, T, \) and \( M \) collectively exhaustive? Yes. In fact \( N \) and \( T \) are collectively exhaustive (\( N U T = S \)) so \( M \) is irrelevant.

(c) Are \( T \) and \( O \) mutually exclusive?
Yes. \( T \cap O = \emptyset \).

(d) Can I order a Tuscan pizza w/ mushrooms and onions?
No. \( O \cap T = \emptyset \).

(e) Can I order a Neapolitan pizza w/ neither mushrooms nor onions? Yes \( \neg M \cap \neg O \neq \emptyset \).
1.2.2 3 machines X, Y, and Z produce 1 circuit. Each circuit is accepted (A) or fails (F). The quality control (QC) test tests one from each machine, and represents the observation as one of \{aaa, aaf, afa, aff, faa, faf, ffa, fff\} = S \ (a)

(b) \(Z_f = \{\text{circuit from } Z \text{ fails}\} = \{aaf, aff, faf, fff\}\)

(check: if Z fails we have two other binary options so \(|Z_f| \text{ should be } 2^2 = 4\})

\(X_a = \{\text{circuit from } X \text{ is acceptable}\} = \{aaa, aaf, afa, aff\}\)

(same size as \(Z_f\))

(c) Are \(Z_f\) and \(X_a\) mutually exclusive? No. \(Z_f \cap X_a = \{aaf, aff\}\)

(d) Are \(Z_f\) and \(X_a\) collectively exhaustive? No. Each has 4 elements and their intersection contains 2 elements, so their union contains only 6 of the 8 els of \(S\)
(e) \( C = \{ \text{more than one circuit acceptable} \} \)
\[ C = \{ qaf, afa, Faa, aac \} \]
\[ |C| = |C| = \binom{3}{2} + \binom{3}{3} = 3 + 1 = 4 \]

\( D = \{ \text{at least 2 circuits fail} \} \)
\[ D = \{ ffa, faF, \ldots, aff, fff \} \]
\[ |D| = |D| = \binom{3}{2} + \binom{3}{3} = 3 + 1 = 4 \]

(F) Are C and D Mutually Exclusive?

Yes. Can't have more than 1 a and 2 f in 3 circuits.

\( C \cap D = \emptyset \)

(g) Are C and D Collectively Exhausting?

Yes. \( C \cup D = \emptyset \) so \(|C \cup D| = 8\)

And \(|S| = 8\).
1.2.4 Let the experiment consist of determining the birth day of an arbitrary individual. By "day," we mean to ignore the year.

(a) Find the sample space. The sample space must include all days of a leap year. The days can be determined from the rhyme:

"Thirty days hath September,
April, June, and November,
All the rest have thirty-one,
Except February, which has 28 or 29..."

(b) How many outcomes are in July? 31 as above.

[Box]
1.2.6: The sample space consists of pairs of non-negative values.

Four give examples of partitions.

Of course there are infinitely many possibilities. Here are 4:

1) \[ S_1 = \{ (x, y) : x = 0 \} \]
   \[ S_2 = \{ (x, y) : x > 0 \} \]

2) \[ S_1 = \{ (x, y) : x < y \} \]
   \[ S_2 = \{ (x, y) : x \geq y \} \]

3) \[ S_1 = \{ (x, y) : xy > 2 \} \]
   \[ S_2 = \{ (x, y) : xy \leq 2 \} \]

4) \[ S_1 = \{ (x, y) : x < x_0, y < y_0, \frac{x_0, y_0 > 0}{\text{fixed numbers}} \} \]
   \[ S_2 = \{ (x, y) : x \geq x_0 \text{ or } y \geq y_0 \} \]
1.3.2] Given 2 fair 6-sided dice, one red, one white, are rolled once.
Let $R_i$ be the event that the red die lands with $i$ face up.
Let $W_i$ be defined similarly.

(a) Find $P[R_3 W_2].$

$$P[R_3 W_2] = \frac{1}{6} \left( \frac{1}{6} \right) = \frac{1}{36}$$

R_i, W_i independent

(b) Let $S_i$ be the event that the two values sum to $i$. Then

$$P[S_5] = P[R_1 W_4 \lor R_2 W_3 \lor R_3 W_2 \lor R_4 W_1]$$

$$= P[R_1 W_4] + P[R_2 W_3] + P[R_3 W_2] + P[R_4 W_1] = 4 \left( \frac{1}{36} \right) = \frac{1}{9}$$
TRUE OR FALSE:

(a) "IF $P(A) = 2P(A^c)$ THEN $P(A) = \frac{1}{2}$.
FALSE: $A \cup A^c = \emptyset$, $P(A) + P(A^c) = 1$
THEN $P(A) = 2P(A^c) = 2(1 - P(A))$
SO $3P(A) = 2$, SO $P(A) = \frac{2}{3}$
AND $2P(A^c) = 2(\frac{1}{3})$.

(b) "FOR ALL $A, B$, $P(AB) \leq P(A)P(B)$".
FALSE: LET $A = B$. THEN FOR
$P(A) = P(B)$ IN $(0, 1)$
$P(AB) = P(A) = P(B) > P(A)P(B)$.
IF WE CHOOSE $A$ AND $B$ S.T.
$P(A) = 0$ OR $P(A) = 1$ THE
STATEMENT HOLDS, BUT WE
NEED ONLY ONE COUNTER-
EXAMPLE TO PROVE IT FALSE.

(c) "IF $P(A) < P(B)$ THEN $P(AB) < P(B)$".
TRUE: $AB = A$ SO
$P(AB) \leq P(A) < P(B)$.

(d) "IF $P(AB|B) = P(A)$, THEN $P(A) \geq P(B)$."
FALSE: \[ A = (A \setminus B) \cup (A \cap B) \]

so \( P[A] = P[A \setminus B] + P[A \cap B] \)

then if \( P[A] = P[A \cap B], \ P[A \setminus B] = 0 \)

\[ B = (B \setminus A) \cup (A \cap B) \]

so \( P[B] = P[B \setminus A] + P[A \cap B] \)

and as long as \( B \setminus A \) is not empty and has \( P[B \setminus A] > 0 \) then

\[ P[B] > A. \]

Alternatively, if \( A = \emptyset \) and \( P[B] > 0 \)
then \( P[A \cap B] = \emptyset \) so \( P[A \cup B] = P[A] = 0 < P[B] \)

[]
Given: Cell phones are either handheld (H) or vehicle mounted (M) and move either fast (F) or slow (W).

\[ P[F] = \frac{1}{2}; \quad P[HF] = \frac{1}{5}; \quad P[MW] = \frac{1}{6}. \]

(a) What is the sample space? 
\[ S = \{ HF, HW, MF, MW \} \]

(b) Find \( P[W] \).
\[ P[W] = 1 - P[F] = \frac{1}{2} \]

(c) Find \( P[MF] \).
\[ P[MF] = P[F] - P[HF] \]
\[ = \frac{1}{2} - \frac{1}{5} = \frac{5 - 2}{10} = \frac{3}{10} \]

(d) Find \( P[H] \).
\[ P[H] = P[HF] + P[HW] \]
\[ P[HW] = P[W] - P[MW] \]
so \[ P[H] = P[HF] + P[MW] + \left( 1 - \frac{P[F]}{P[W]} \right) \]
\[ = \frac{1}{5} - \frac{1}{10} + \frac{1}{2} \]
\[ = \frac{2 - 1 + 5}{10} = \frac{3}{5} \]
[1.3.8] GIVEN: ONE SIX-SIDED FAIR DIE ROLLED ONCE.

(a) WHAT IS THE SAMPLE SPACE?
   \[ S = \{1, 2, 3, 4, 5, 6\} \]

(b) WHAT IS THE PROBABILITY OF EACH OUTCOME?
   THE SIDES OF THE FAIR DIE
   LANDING FACE-UP ARE EQUALLY PROBABLE SO \( P(\text{s} \in S) = \frac{1}{|S|} = \frac{1}{6} \)

(c) LET \( E \) BE THE EVENT THE ROLL IS EVEN. FIND \( P(E) \).
   THE OUTCOMES ARE MUTUALLY EXCLUSIVE SO
   \[ P(E) = P(2) + P(4) + P(6) = \frac{3}{6} = \frac{1}{2} \]
1.3.10. Use Thm 1.4 to prove the following:

(a) $P[A \cup B] \geq P[A].$

   $A \subseteq A \cup B$ so by Thm 1.4d
   $P[A \cup B] \geq P[A]$

(b) $P[A \cup B] \geq P[B]$

   Again Thm 1.4d, mutatis mutandis

(c) $P[A \cap B] \leq P[A]$  

   $A \cap B \subseteq A$ so true by Thm 1.4d

(d) $P[A \cap B] \leq P[B]$  

   $A \cap B \subseteq B$ so by Thm 1.4d, the result follows.

□
Using only the three axioms of probability, prove $P(\emptyset) = 0$.

We will need Lemma

If $A_1$ and $A_2$ are mut. excl. then $P(A_1 U A_2) = P(A_1) + P(A_2)$.

Proof

Let $A_3 = A_4 = \ldots = \emptyset$. Then

$A_1 \cap A_{3+k} = \emptyset \ \forall k \geq 0$

$A_2 \cap A_{3+k} = \emptyset \ \forall k \geq 0$

$P(\bigcup_{i=1}^{\infty} A_i) = P(A_1) + P(A_2) + \sum_{i=3}^{\infty} P(\emptyset)$

"$A_1 U A_2$ since $A_3, A_4, \ldots$ are empty.

Thus $P(A_1 U A_2) = P(A_1) + P(A_2) + \sum_{i=3}^{\infty} P(\emptyset)$

If $P(\emptyset) > 0$ then the sum on the RHS diverges and is not limited by 1, contradicting Axiom 2 $P(S) = 1$ (since $\emptyset \subset S$).

We know that each term of the sum is nonnegative by Axiom 1 so.
Now we have already shown $P[\emptyset] = 0$, but we could also apply the lemma to conclude that $P[S] = P[S] + P[\emptyset]$ so $P[\emptyset] = 0$ since $S \cap \emptyset$ is empty. So $S$ and $\emptyset$ are disjoint.

\[\]
1.3.14 | FOR EACH FACT IN THM 1.4 DETERMINE WHICH AXIOMS ARE REQUIRED TO PROVE IT.

(a) \( P[\varphi] = 0 \).

We used axioms 1, 2 and 3 in the previous problem, but there may be other ways to prove it.

(b) \( P[A^c] = 1 - P[A] \)

Applying axiom 2 with \( S = A^c \cup A \) and our lemma 80 axioms 1, 2 and 3 (but there may be other options).

(c) For any A and B (not nec. mutually exclusive)

\[
\]

\[
P[A] = P[A \setminus B] + P[AB]
\]

By an application of our lemma.

Similarly \( P[B] = P[B \setminus A] + P[AB] \).

Now \( A \cup B = (A \setminus B) \cup (A \cap B) \cup (B \setminus A) \)

So if we create a new lemma for the sum of
THREE MUTUALLY EXCLUSIVE EVENTS, WE HAVE THIS RESULT (i.e. USING AXIOMS 1, 2 AND 3).

(d) IF A ⊆ B THEN \( P(A) \leq P(B) \).

LET \( B = (B \cap A) \cup (B \cap A^c) \).
THE TWO PARTS ARE MUT. EXCL. SO BY OUR LEMMA,
\( P(B) = P(B \cap A) \cup P(B \cap A^c) \)

BUT IF A ⊆ B THEN A = B \cap A.
BY AXIOM 1 \( P(B \cap A^c) > 0 \)
SO \( P(A) \leq P(B) \). AGAIN
WE USED ALL 3 AXIOMS BUT
THAT DOES NOT MEAN THE
STATEMENT CANNOT BE
PROVED WITH FEWER.