Let $s_i$ denote the outcome that the roll is $i$. So, for $1 \leq i \leq 6$, $R_i = \{s_i\}$. Similarly, $G_j = \{s_{j+1}, \ldots, s_6\}$. Note that $P[R_iG_j] = P[R_i]$ if $i > j$ and 0 otherwise. $P[R_i] = 1/6$ for all $i$. $P[E] = 1/2$. $P[G_j] = (6-i)/6$.

(a) Since $G_1 = \{s_2, s_3, s_4, s_5, s_6\}$ and all outcomes have probability $1/6$, $P[G_1] = 5/6$. The event $R_3G_1 = \{s_3\}$ and $P[R_3G_1] = 1/6$ so that

$$P[R_3|G_1] = \frac{P[R_3G_1]}{P[G_1]} = \frac{1}{5}. \quad (1)$$

(b) The conditional probability that 6 is rolled given that the roll is greater than 3 is

$$P[R_6|G_3] = \frac{P[R_6G_3]}{P[G_3]} = \frac{P[s_6]}{P[s_4, s_5, s_6]} = \frac{1/6}{3/6} = \frac{1}{3}. \quad (2)$$

(c) The event $E$ that the roll is even is $E = \{s_2, s_4, s_6\}$ and has probability $3/6$. The joint probability of $G_3$ and $E$ is

$$P[G_3E] = P[s_4, s_6] = 1/3. \quad (3)$$

The conditional probabilities of $G_3$ given $E$ is

$$P[G_3|E] = \frac{P[G_3E]}{P[E]} = \frac{1/3}{1/2} = \frac{2}{3}. \quad (4)$$

(d) The conditional probability that the roll is even given that it’s greater than 3 is

$$P[E|G_3] = \frac{P[EG_3]}{P[G_3]} = \frac{1/3}{1/2} = \frac{2}{3}. \quad (5)$$
Prob 1.4.4 Solution

Let $A_i$ and $B_i$ denote the events that the $i$th phone sold is an Apricot or a Banana respectively. Our goal is to find $P[B_1B_2]$, but since it is not clear where to start, we should plan on filling in the table

<table>
<thead>
<tr>
<th></th>
<th>$A_1$</th>
<th>$B_1$</th>
<th>$A_2$</th>
<th>$B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>$P[A_1A_2]$</td>
<td>$P[A_1B_2]$</td>
<td>$P[A_2A_2]$</td>
<td>$P[A_2B_2]$</td>
</tr>
<tr>
<td>$B_1$</td>
<td>$P[B_1A_2]$</td>
<td>$P[B_1B_2]$</td>
<td>$P[B_2A_2]$</td>
<td>$P[B_2B_2]$</td>
</tr>
</tbody>
</table>

This table has four unknowns: $P[A_1A_2]$, $P[A_1B_2]$, $P[B_1A_2]$, and $P[B_1B_2]$. We start knowing that

$$P[A_1A_2] + P[A_1B_2] + P[B_1A_2] + P[B_1B_2] = 1. \tag{1}$$

We still need three more equations to solve for the four unknowns. From “sales of Apricots and Bananas are equally likely,” we know that $P[A_i] = P[B_i] = 1/2$ for $i = 1, 2$. This implies

$$P[A_1] = P[A_1A_2] + P[A_1B_2] = 1/2, \tag{2}$$

$$P[A_2] = P[A_1A_2] + P[B_1A_2] = 1/2. \tag{3}$$

The final equation comes from “given that the first phone sold is a Banana, the second phone is twice as likely to be a Banana,” which implies $P[B_2|B_1] = 2P[A_2|B_1]$. Using Bayes’ theorem, we have

$$\frac{P[B_1B_2]}{P[B_1]} = 2 \frac{P[B_1A_2]}{P[B_1]} \implies P[B_1A_2] = \frac{1}{2} P[B_1B_2]. \tag{4}$$

Replacing $P[B_1A_2]$ with $P[B_1B_2]/2$ in the the first three equations yields

$$P[A_1A_2] + P[A_1B_2] + \frac{3}{2} P[B_1B_2] = 1, \tag{5}$$

$$P[A_1A_2] + P[A_1B_2] = 1/2, \tag{6}$$

$$P[A_1A_2] + \frac{1}{2} P[B_1B_2] = 1/2. \tag{7}$$

Subtracting (6) from (5) yields $(3/2)P[B_1B_2] = 1/2$, or $P[B_1B_2] = 1/3$, which is the answer we are looking for.
At this point, if you are curious, we can solve for the rest of the probability table. From (4), we have $P[B_1A_2] = 1/6$ and from (7) we obtain $P[A_1A_2] = 1/3$. It then follows from (6) that $P[A_1B_2] = 1/6$. The probability table is

<table>
<thead>
<tr>
<th></th>
<th>$A_2$</th>
<th>$B_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>1/3</td>
<td>1/6</td>
</tr>
<tr>
<td>$B_1$</td>
<td>1/6</td>
<td>1/3</td>
</tr>
</tbody>
</table>

**Problem 1.4.5 Solution**

The first generation consists of two plants each with genotype $yg$ or $gy$. They are crossed to produce the following second generation genotypes, $S = \{yy, yg, gy, gg\}$. Each genotype is just as likely as any other so the probability of each genotype is consequently 1/4. A pea plant has yellow seeds if it possesses at least one dominant $y$ gene. The set of pea plants with yellow seeds is

$$Y = \{yy, yg, gy\}.$$  \hspace{1cm} (1)

So the probability of a pea plant with yellow seeds is

$$P[Y] = P[yy] + P[yg] + P[gy] = 3/4.$$  \hspace{1cm} (2)

**Problem 1.4.6 Solution**

Define $D$ as the event that a pea plant has two dominant $y$ genes. To find the conditional probability of $D$ given the event $Y$, corresponding to a plant having yellow seeds, we look to evaluate

$$P[D|Y] = \frac{P[DY]}{P[Y]}.$$  \hspace{1cm} (1)

Note that $P[DY]$ is just the probability of the genotype $yy$. From Problem 1.4.5, we found that with respect to the color of the peas, the genotypes $yy$, $yg$, $gy$, and $gg$ were all equally likely. This implies

$$P[DY] = P[yy] = 1/4 \quad P[Y] = P[yy, gy, yg] = 3/4.$$  \hspace{1cm} (2)

Thus, the conditional probability can be expressed as

$$P[D|Y] = \frac{P[DY]}{P[Y]} = \frac{1/4}{3/4} = 1/3.$$  \hspace{1cm} (3)
Problem 1.4.8 Solution

The problem statement yields the obvious facts that \(P[L] = 0.16\) and \(P[H] = 0.10\). The words “10% of the ticks that had either Lyme disease or HGE carried both diseases” can be written as

\[
P[LH | L \cup H] = 0.10. \tag{1}
\]

(a) Since \(LH \subset L \cup H\),

\[
P[LH | L \cup H] = \frac{P[LH \cap (L \cup H)]}{P[L \cup H]} = \frac{P[LH]}{P[L \cup H]} = 0.10. \tag{2}
\]

Thus,

\[
P[LH] = 0.10 P[L \cup H] = 0.10 (P[L] + P[H] - P[LH]). \tag{3}
\]

Since \(P[L] = 0.16\) and \(P[H] = 0.10\),

\[
P[LH] = \frac{0.10 (0.16 + 0.10)}{1.1} = 0.0236. = 13/50 \tag{4}
\]

(b) The conditional probability that a tick has HGE given that it has Lyme disease is

\[
P[H | L] = \frac{P[LH]}{P[L]} = \frac{0.0236}{0.16} = 0.1475. = 13/88 \tag{5}
\]
Problem 1.5.2 Solution

(a) From the given probability distribution of billed minutes, $M$, the probability that a call is billed for more than 3 minutes is

$$P[L] = 1 - P[3 \text{ or fewer billed minutes}] = 1 - P[B_1] - P[B_2] - P[B_3] = 1 - \alpha - \alpha(1 - \alpha) - \alpha(1 - \alpha)^2 = (1 - \alpha)^3 = 0.57.$$  \hfill (1)

(b) The probability that a call will billed for 9 minutes or less is

$$P[9 \text{ minutes or less}] = \sum_{i=1}^{9} \alpha(1 - \alpha)^{i-1} = 1 - (0.57)^3.  \hfill (2)$$
Problem 1.6.1 Solution

This problem asks whether \( A \) and \( B \) can be independent events yet satisfy \( A = B \)?

By definition, events \( A \) and \( B \) are independent if and only if \( P[AB] = P[A]P[B] \).

We can see that if \( A = B \), that is they are the same set, then
\[
\]

Thus, for \( A \) and \( B \) to be the same set and also independent,
\[
\]

There are two ways that this requirement can be satisfied:

- \( P[A] = 1 \) implying \( A = B = S \).
- \( P[A] = 0 \) implying \( A = B = \emptyset \).

Problem 1.6.2 Solution

From the problem statement, we learn three facts:

\[
\begin{align*}
P[AB] &= 0 \quad \text{(since} \ A \text{ and} \ B \text{ are mutually exclusive)} \quad (1) \\
P[AB] &= P[A]P[B] \quad \text{(since} \ A \text{ and} \ B \text{ are independent)} \quad (2) \\
P[A] &= P[B] \quad \text{(since} \ A \text{ and} \ B \text{ are equiprobable)} \quad (3)
\end{align*}
\]

Applying these facts in the given order, we see that
\[
0 = P[AB] = P[A]P[B] = (P[A])^2. \quad (4)
\]

It follows that \( P[A] = 0 \).

The 3rd "=" is important here because \( P[AB]=0 \) implies that at least one of \( P[A] \) and \( P[B]=0 \) but not that both are zero.

Problem 1.6.3 Solution

Let \( A_i \) and \( B_i \) denote the events that the \( i \)th phone sold is an Apricot or a Banana respectively. The works "each phone sold is twice as likely to be an Apricot than a Banana" tells us that
\[
P[A_i] = 2P[B_i].
\]

The 3rd "=" is important here because \( P[AB]=0 \) implies that at least one of \( P[A] \) and \( P[B]=0 \) but not that both are zero.
Problem 1.6.4 Solution

In the Venn diagram, assume the sample space has area 1 corresponding to probability 1. As drawn, both $A$ and $B$ have area $1/4$ so that $P[A] = P[B] = 1/4$. Moreover, the intersection $AB$ has area $1/16$ and covers $1/4$ of $A$ and $1/4$ of $B$. That is, $A$ and $B$ are independent since

$$P[AB] = P[A]P[B].$$  \hspace{1cm} (1)
Problem 1.6.6 Solution  Given \( P[C] = 5/8 \), \( P[D] = 3/8 \), \( C, D \) independent.

(a) Since \( C \) and \( D \) are independent,

\[
P[C \cap D] = P[C] P[D] = 15/64. \tag{1}
\]

The next few items are a little trickier. From Venn diagrams, we see

Don't need to draw the Venn diagram.

\[
P[C \cap D^c] = P[C] - P[C \cap D] = 5/8 - 15/64 = 25/64. \tag{2}
\]

It follows that

\[
P[C \cup D^c] = P[C] + P[D^c] - P[C \cap D^c] \tag{3}
\]
\[
= 5/8 + (1 - 3/8) - 25/64 = 55/64. \tag{4}
\]

Using DeMorgan's law, we have because \( P[C \text{ or } D] = P[C] + P[D] - P[CD] \)

\[
P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 15/64. \tag{5}
\]

(b) Since \( P[C^c D^c] = P[C^c] P[D^c] \), \( C^c \) and \( D^c \) are independent. (by the definition of independence)
Problem 1.6.8 Solution  \[ \text{Given C, D independent, } P[CD]=1/3, P[C]=1/2 \]

(a) Since \( C \) and \( D \) are independent \( P[CD] = P[C]P[D] \). So

\[
P[D] = \frac{P[CD]}{P[C]} = \frac{1/3}{1/2} = 2/3. \tag{1}
\]

In addition, \( P[C \cap D^c] = P[C] - P[C \cap D] = 1/2 - 1/3 = 1/6 \). To find \( P[C^c \cap D^c] \), we first observe that

\[
P[C \cup D] = P[C] + P[D] - P[C \cap D] = 1/2 + 2/3 - 1/3 = 5/6. \tag{2}
\]

By De Morgan’s Law, \( C^c \cap D^c = (C \cup D)^c \). This implies

\[
P[C^c \cap D^c] = P[(C \cup D)^c] = 1 - P[C \cup D] = 1/6. \tag{3}
\]

Note that a second way to find \( P[C^c \cap D^c] \) is to use the fact that if \( C \) and \( D \) are independent, then \( C^c \) and \( D^c \) are independent. Thus

\[
P[C^c \cap D^c] = P[C^c]P[D^c] = (1 - P[C])(1 - P[D]) = 1/6. \tag{4}
\]

Finally, since \( C \) and \( D \) are independent events, \( P[C|D] = P[C] = 1/2 \).

(b) Note that we found \( P[C \cup D] = 5/6 \). We can also use the earlier results to show

\[
P[C \cup D^c] = P[C] + P[D] - P[C \cap D^c]
= 1/2 + (1 - 2/3) - 1/6 = 2/3. \tag{6}
\]

(c) By Definition 1.6, events \( C \) and \( D^c \) are independent because

\[
P[C \cap D^c] = 1/6 = (1/2)(1/3) = P[C]P[D^c]. \tag{7}
\]
Problem 1.6.10 Solution

There are 16 distinct equally likely outcomes for the second generation of pea plants based on a first generation of \(\{rwyg, rwgy, wryg, wrgy\}\). These are:

\[
\begin{align*}
rryy & \quad rryg & \quad rrgy & \quad rrgg \\
rwyy & \quad rwyg & \quad rwgy & \quad rwgg \\
wryy & \quad wryg & \quad wrgy & \quad wrgg \\
wwyy & \quad wwyg & \quad wwgy & \quad wwgg
\end{align*}
\]

which yield: RY, RY, RY, RG (first 3 rows) and WY, WY, WY, WG


A plant has yellow seeds, that is event \(Y\) occurs, if a plant has at least one dominant \(y\) gene. Except for the four outcomes with a pair of recessive \(g\) genes, the remaining 12 outcomes have yellow seeds. From the above, we see that

\[
P[Y] = 12/16 = 3/4 \quad (1)
\]

and

\[
P[R] = 12/16 = 3/4. \quad (2)
\]

To find the conditional probabilities \(P[R|Y]\) and \(P[Y|R]\), we first must find \(P[RY]\). Note that \(RY\), the event that a plant has rounded yellow seeds, is the set of outcomes

\[
RY = \{rryy, rryg, rrgy, rwyy, rwyg, rwgy, wryy, wryg, wrgy\}. \quad (3)
\]
Since $P[RY] = 9/16$,

$$P[Y|R] = \frac{P[RY]}{P[R]} = \frac{9/16}{3/4} = 3/4$$

and

$$P[R|Y] = \frac{P[RY]}{P[Y]} = \frac{9/16}{3/4} = 3/4.$$  (4)

Thus $P[R|Y] = P[R]$ and $P[Y|R] = P[Y]$ and $R$ and $Y$ are independent events.

There are four visibly different pea plants, corresponding to whether the peas are round ($R$) or not ($R^c$), or yellow ($Y$) or not ($Y^c$). These four visible events have probabilities

$$P[RY] = 9/16 \quad P[RY^c] = 3/16,$$  (6)

$$P[R^cY] = 3/16 \quad P[R^cY^c] = 1/16.$$  (7)
Problem 1.6.12 Solution

In the Venn diagram at right, assume the sample space has area 1 corresponding to probability 1. As drawn, $A$, $B$, and $C$ each have area $1/2$ and thus probability $1/2$. Moreover, the three way intersection $ABC$ has probability $1/8$. Thus $A$, $B$, and $C$ are mutually independent since


(1)