Solutions to HW6

Note: Most of these solutions were generated by R. D. Yates and D. J. Goodman, the authors of our textbook. I have added comments in italics where I thought more detail was appropriate. Those solutions I have written myself are designated by my initials.

Problem 4.2.2 ●
The cumulative distribution function of the continuous random variable $V$ is

$$F_V(v) = \begin{cases} 
0 & v < -5, \\
c(v + 5)^2 & -5 \leq v < 7, \\
1 & v \geq 7.
\end{cases}$$

(a) What is $c$?
(b) What is $P[V > 4]$?
(c) $P[-3 < V \leq 0]$?
(d) What is the value of $a$ such that $P[V > a] = 2/3$?

Problem 4.2.2 Solution

The CDF of $V$ was given to be

$$F_V(v) = \begin{cases} 
0 & v < -5, \\
c(v + 5)^2 & -5 \leq v < 7, \\
1 & v \geq 7.
\end{cases} \quad (1)$$

(a) For $V$ to be a continuous random variable, $F_V(v)$ must be a continuous function. This occurs if we choose $c$ such that $F_V(v)$ doesn’t have a discontinuity at $v = 7$. We meet this requirement if $(c(7 + 5)^2 = 1$. This implies $c = 1/144$.

(b) $P[V > 4] = 1 - P[V \leq 4] = 1 - F_V(4) = 1 - 81/144 = 63/144 \quad (2)$

(c) $P[-3 < V \leq 0] = F_V(0) - F_V(-3) = 25/144 - 4/144 = 21/144 \quad (3)$

(d) Since $0 \leq F_V(v) \leq 1$ and since $F_V(v)$ is a nondecreasing function, it must be that $-5 \leq a \leq 7$. In this range,

$$P[V > a] = 1 - F_V(a) = 1 - (a + 5)^2/144 = 2/3 \quad (4)$$

The unique solution in the range $-5 \leq a \leq 7$ is $a = 4\sqrt{3} - 5 = 1.928$. 
Problem 4.3.4 ■
For a constant parameter \( a > 0 \), a Rayleigh random variable \( X \) has PDF
\[
x f_X(x) = \begin{cases} 
  a^2 x e^{-a^2 x^2 / 2} & x > 0, \\
  0 & \text{otherwise}.
\end{cases}
\]

What is the CDF of \( X \)?

Problem 4.3.4 Solution
For \( x < 0 \), \( F_X(x) = 0 \). For \( x \geq 0 \),
\[
F_X(x) = \int_0^x f_X(y) \, dy 
\]
\[
= \int_0^x a^2 y e^{-a^2 y^2 / 2} \, dy
\]
\[
= -e^{-a^2 y^2 / 2} \bigg|_0^x = 1 - e^{-a^2 x^2 / 2}
\]

A complete expression for the CDF of \( X \) is
\[
F_X(x) = \begin{cases} 
  0 & x < 0 \\
  1 - e^{-a^2 x^2 / 2} & x \geq 0
\end{cases}
\]

Problem 4.3.6 ♦♦
For constants \( a \) and \( b \), random variable \( X \) has PDF
\[
x f_X(x) = \begin{cases} 
  a x^2 + b x & 0 \leq x \leq 1, \\
  0 & \text{otherwise}.
\end{cases}
\]

What conditions on \( a \) and \( b \) are necessary and sufficient to guarantee that \( f_X(x) \) is a valid PDF?

Problem 4.3.6 Solution
\[
x f_X(x) = \begin{cases} 
  a x^2 + b x & 0 \leq x \leq 1 \\
  0 & \text{otherwise}
\end{cases}
\]

First, we note that \( a \) and \( b \) must be chosen such that the above PDF integrates to 1.
\[
\int_0^1 (ax^2 + bx) \, dx = a/3 + b/2 = 1
\]

Hence, \( b = 2 - 2a/3 \) and our PDF becomes
\[
x f_X(x) = x(ax + 2 - 2a/3)
\]
For the PDF to be non-negative for \( x \in [0, 1] \), we must have \( ax + 2 - 2a/3 \geq 0 \) for all \( x \in [0, 1] \). This requirement can be written as

\[ a(2/3 - x) \leq 2 \quad (0 \leq x \leq 1) \quad (4) \]

For \( x = 2/3 \), the requirement holds for all \( a \). However, the problem is tricky because we must consider the cases \( 0 \leq x < 2/3 \) and \( 2/3 < x \leq 1 \) separately because of the sign change of the inequality. When \( 0 \leq x < 2/3 \), we have \( 2/3 - x > 0 \) and the requirement is most stringent at \( x = 0 \) where we require \( 2a/3 \leq 2 \) or \( a \leq 3 \). When \( 2/3 < x \leq 1 \), we can write the constraint as \( a(x - 2/3) \geq -2. \) In this case, the constraint is most stringent at \( x = 1 \), where we must have \( a/3 \geq -2 \) or \( a \geq -6 \). Thus a complete expression for our requirements are

\[ -6 \leq a \leq 3 \quad b = 2 - 2a/3 \quad (5) \]

As we see in the following plot, the shape of the PDF \( f_X(x) \) varies greatly with the value of \( a \).

### Problem 4.4.2 Solution (SK)

Let \( X \) be a continuous random variable with PDT

\[ f_X(x) = \begin{cases} 1/8 & 1 \leq x \leq 9, \\ 0 & \text{otherwise}. \end{cases} \quad (6) \]

Let \( Y = h(X) = 1/\sqrt{X} \).

(a) Find \( \mathbb{E}[X] \) and \( \text{Var}[X] \).

\[
\mathbb{E}[X] = \int_1^9 x f_X(x) dx = \int_1^9 x \left( \frac{1}{8} \right) dx = \left( \frac{1}{8} \right) \left[ \frac{x^2}{2} \right]_1^9 = \frac{1}{16} (81 - 1) = 5, \quad (7) \\
\mathbb{E}[X^2] = \int_1^9 x^2 f_X(x) dx = \int_1^9 x^2 \left( \frac{1}{8} \right) dx = \left[ \frac{x^3}{24} \right]_1^9 = \frac{1}{24} (729 - 1) = 91/3, \quad (8)
\]

so

\[
\text{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 91/3 - 25 = (91 - 75)/3 = 16/3, \quad (9)
\]
(b) Find \( h(E[X]) \) and \( E[h(X)] \).

\[
E[h(X)] = h(5) = \frac{1}{\sqrt{5}}. \tag{10}
\]

\[
E[h(X)] = \int_1^9 \frac{1}{\sqrt{x}} f_X(x) \, dx = \frac{1}{8} x^{-1/2} \, dx = 2 x^{1/2} \frac{1}{8} \bigg|_1^9 = \frac{1}{4} (3 - 1) = 1/2. \tag{11}
\]

(c) Find \( E[Y] \) and \( \text{Var}[Y] \).

From above, \( E[Y] = 1/2 \).

\[
E[Y^2] = E[(h(X))^2] = \int_{-5}^7 \frac{1}{8} x \, dx = \frac{1}{8} \ln x \bigg|_9^1 = \frac{1}{8} (\ln 9 - \ln 1) = \frac{1}{8} \ln 9. \tag{12}
\]

\[
\text{Var}[Y] = E[Y^2] - (E[Y])^2 = \frac{1}{8} \ln 9 - 1/4 \tag{13}
\]

**Problem 4.4.6**

The cumulative distribution function of random variable \( V \) is

\[
F_V(v) = \begin{cases} 
0 & v < -5, \\
(v + 5)^2/144 & -5 \leq v < 7, \\
1 & v \geq 7.
\end{cases}
\]

(a) What is \( E[V] \)?

(b) What is \( \text{Var}[V] \)?

(c) What is \( E[V^3] \)?

**Problem 4.4.6 Solution**

To evaluate the moments of \( V \), we need the PDF \( f_V(v) \), which we find by taking the derivative of the CDF \( F_V(v) \). The CDF and corresponding PDF of \( V \) are

\[
F_V(v) = \begin{cases} 
0 & v < -5, \\
(v + 5)^2/144 & -5 \leq v < 7, \\
1 & v \geq 7
\end{cases} \quad f_V(v) = \begin{cases} 
0 & v < -5, \\
(v + 5)/72 & -5 \leq v < 7, \\
0 & v \geq 7
\end{cases} \tag{1}
\]

(a) The expected value of \( V \) is

\[
E[V] = \int_{-\infty}^{\infty} v f_V(v) \, dv = \frac{1}{72} \int_{-5}^{7} (v^2 + 5v) \, dv \tag{2}
\]

\[
= \frac{1}{72} \left( \frac{v^3}{3} + \frac{5v^2}{2} \right) \bigg|_{-5}^{7} = \frac{1}{72} \left( \frac{343}{3} + \frac{245}{2} + \frac{125}{3} - \frac{125}{2} \right) = 3 \tag{3}
\]
(b) To find the variance, we first find the second moment

\[
E [V^2] = \int_{-\infty}^{\infty} v^2 f_V (v) \, dv = \frac{1}{72} \int_{-5}^{7} (v^3 + 5v^2) \, dv = 1 \frac{719}{432} = 15.55
\]

The variance is \( \text{Var}[V] = E[V^2] - (E[V])^2 = 2831/432 = 6.55. \)

(c) The third moment of \( V \) is

\[
E [V^3] = \int_{-\infty}^{\infty} v^3 f_V (v) \, dv = \frac{1}{72} \int_{-5}^{7} (v^4 + 5v^3) \, dv = 1 \frac{86.2}{5} = 86.2
\]

**Problem 4.5.4**

\( Y \) is an exponential random variable with variance \( \text{Var}[Y] = 25. \)

(a) What is the PDF of \( Y \)?

(b) What is \( E[Y^2] \)?

(c) What is \( P[Y > 5] \)?

**Problem 4.5.4 Solution**

From Appendix A, we observe that an exponential PDF \( Y \) with parameter \( \lambda > 0 \) has PDF

\[
f_Y (y) = \begin{cases} 
\lambda e^{-\lambda y} & y \geq 0 \\
0 & \text{otherwise}
\end{cases}
\]

In addition, the mean and variance of \( Y \) are

\[
E [Y] = \frac{1}{\lambda} \quad \text{Var}[Y] = \frac{1}{\lambda^2}
\]

(a) Since \( \text{Var}[Y] = 25 \), we must have \( \lambda = 1/5. \)

(b) The expected value of \( Y \) is \( E[Y] = 1/\lambda = 5. \)  
Completing the solution, since

\[
\text{Var}Y = E[Y^2] - (E[Y])^2,
\]

\[
E[Y^2] = \text{Var}[Y] + (E[Y])^2 = 25 + 5^2 = 50.
\]
Of course it is also possible to calculate $E[Y^2]$ directly:

$$E[Y^2] = \int_0^\infty y^2 \lambda e^{-\lambda y} dy = \left[-y^2 e^{-\lambda y}\right]_0^\infty - \int_0^\infty -2y e^{-\lambda y} dy$$

$$= (0 - 0) + \int_0^\infty 2y e^{-\lambda y} dy$$

$$= -\left[\frac{2y}{\lambda} e^{-\lambda y}\right]_0^\infty - \int_0^\infty -2e^{-\lambda y} dy$$

$$= (0 - 0) + \frac{2}{y^2} e^{-\lambda y} \bigg|_0^\infty = \frac{2}{(1/5)^2} = 50$$

where the first integration by parts uses

$$u = y^2 \quad dv = \lambda e^{-\lambda y} dy$$

$$du = 2ydy \quad v = -e^{-\lambda y}$$

and the second uses

$$u = 2y \quad dv = e^{-\lambda y} dy$$

$$du = 2dy \quad v = -\frac{e^{-\lambda y}}{\lambda}.$$  

(SK)

(c)

$$P[Y > 5] = \int_5^\infty f_Y(y) dy = -e^{-y/5} \bigg|_5^\infty = e^{-1} \quad (3)$$

Problem 4.5.6

$X$ is an Erlang $(n, \lambda)$ random variable with parameter $\lambda = 1/3$ and expected value $E[X] = 15$.

(a) What is the value of the parameter $n$?

(b) What is the PDF of $X$?

(c) What is $\text{Var}[X]$?

Problem 4.5.6 Solution

From Appendix A, an Erlang random variable $X$ with parameters $\lambda > 0$ and $n$ has PDF

$$f_X(x) = \begin{cases} \lambda^n x^{n-1} e^{-\lambda x} / (n-1)! & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (1)$$

In addition, the mean and variance of $X$ are

$$E[X] = \frac{n}{\lambda} \quad \text{Var}[X] = \frac{n}{\lambda^2} \quad (2)$$
(a) Since $\lambda = 1/3$ and $E[X] = n/\lambda = 15$, we must have $n = 5$.

(b) Substituting the parameters $n = 5$ and $\lambda = 1/3$ into the given PDF, we obtain

$$f_X(x) = \begin{cases} 
(1/3)^5 x^4 e^{-x/3}/24 & x \geq 0 \\
0 & \text{otherwise}
\end{cases}$$

(c) From above, we know that $\text{Var}[X] = n/\lambda^2 = 45$. 

Problem 4.5.12

X is a uniform random variable with expected value $\mu_X = 7$ and variance $\text{Var}[X] = 3$. What is the PDF of $X$?

Problem 4.5.12 Solution

We know that $X$ has a uniform PDF over $[a,b)$ and has mean $\mu_X = 7$ and variance $\text{Var}[X] = 3$. All that is left to do is determine the values of the constants $a$ and $b$, to complete the model of the uniform PDF.

$$E[X] = \frac{a+b}{2} = 7 \quad \text{Var}[X] = \frac{(b-a)^2}{12} = 3$$

(1)

Since we assume $b > a$, this implies

$$a + b = 14 \quad b - a = 6$$

(2)

Solving these two equations, we arrive at

$$a = 4 \quad b = 10$$

(3)

And the resulting PDF of $X$ is,

$$f_X(x) = \begin{cases} 
1/6 & 4 \leq x \leq 10 \\
0 & \text{otherwise}
\end{cases}$$

(4)

Problem 4.6.14 Solution (SK)

At $t = 0$, the stock price is $\$k$.

At $t > 0$, the stock price is $X \sim \mathcal{N}(k,t)$.

At $t$, a Call Option at Strike $k$ has value $V = (X - k)^+$ where the operator $\cdot^+$ is defined as $(z)^+ = \max(z,0)$.

(a) Find the expected value $E[V]$.

Note that since $V = (X - k)^+$, for $X \geq k, dx = dv$.

$$E[V] = \int_{-\infty}^{\infty} v f_V(v) dv = \int_k^{\infty} \frac{(x-k)}{\sqrt{2\pi k}} e^{-(x-k)^2/2t} dx$$

(5)

$$= \frac{1}{\sqrt{2\pi t}} \left( \frac{2t}{2} \right) e^{-(x-k)^2/2t} \bigg|_k^{\infty} = \sqrt{\frac{t}{2\pi}} \left( 0 - e^{-0} \right) = \sqrt{\frac{t}{2\pi}}$$

(6)
(b) Suppose one can buy the call option for \( d \) at \( t = 0 \), sell the call at \( t \) for \( V \), and earn \( R = V - d \). Let \( d_0 \) denote the value of \( d \) for which \( P[R > 0] = 1/2 \). Find \( d_0 \).

By definition of \( R \), \( P[R > 0] = P[V > d] \). By definition of \( V \), \( V \geq 0 \) so for \( d < 0 \), \( P[V > d] = 1 \). Define \( R_0 \) to be the earnings corresponding to using threshold \( d_0 \) for \( V \). Then remembering that \( X \) is Gaussian with mean \( k \) and variance \( t \), and setting \( Z = (X - k)/\sqrt{t} \) we have

\[
P[R_0 > 0] = P[V > d_0] = P[X - k > d_0] = P[Z > d_0/\sqrt{t}] = 1 - \Phi(d_0/\sqrt{t}).
\]

(7) 

\( \Phi(0) = 1/2 \) so \( d_0 = 0 \).

(c) Let \( d_1 \) denote the value of \( d \) for which \( E[R] = 0.01d \). Find \( d_1 \).

\[
E[R] = E[V - d_1] = 0.01d_1 \quad (8)
\]

\[
= \sqrt{\frac{t}{2\pi}} - E[d_1] = 0.01d_1, \quad (9)
\]

so

\[
\sqrt{\frac{t}{2\pi}} = 1.01d_1
\]

and

\[
d_1 = \frac{1}{1.01} \sqrt{\frac{t}{2\pi}}.
\]

(d) Comment on the strategies \( S_1 \) : “Buy the option if \( d \leq d_0 \)” and \( S_2 \) : “Buy the option if \( d \leq d_1 \)”.

If one can find someone who will pay one to take the option, \( S_1 \) is a great strategy. It is, however, unlikely that one will find such a person.

Buying when \( d \leq d_1 \), is a reasonable strategy if \( 0.01d_1 \) will suffice to pay the transaction fee. All one needs to know is \( V \).

Problem 4.7.6

When you make a phone call, the line is busy with probability 0.2 and no one answers with probability 0.3. The random variable \( X \) describes the conversation time (in minutes) of a phone call that is answered. \( X \) is an exponential random variable with \( E[X] = 3 \) minutes. Let the random variable \( W \) denote the conversation time (in seconds) of all calls (\( W = 0 \) when the line is busy or there is no answer.)

(a) What is \( F_W(w) \)?

(b) What is \( f_W(w) \)?

(c) What are \( E[W] \) and \( \text{Var}[W] \)?
Problem 4.7.6 Solution

(a) Since the conversation time cannot be negative, we know that $F_W(w) = 0$ for $w < 0$. The conversation time $W$ is zero iff either the phone is busy, no one answers, or if the conversation time $X$ of a completed call is zero. Let $A$ be the event that the call is answered. Note that the event $A^c$ implies $W = 0$. For $w \geq 0$,

$$F_W(w) = P[A^c] + P[A] F_{W|A}(w) = (1/2) + (1/2) F_X(w)$$

Thus the complete CDF of $W$ is

$$F_W(w) = \begin{cases} 0 & w < 0 \\ 1/2 + (1/2) F_X(w) & w \geq 0 \end{cases}$$

(b) By taking the derivative of $F_W(w)$, the PDF of $W$ is

$$f_W(w) = \begin{cases} (1/2) \delta(w) + (1/2) f_X(w) & \text{if } w \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Next, we keep in mind that since $X$ must be nonnegative, $f_X(x) = 0$ for $x < 0$. Hence,

$$f_W(w) = (1/2) \delta(w) + (1/2) f_X(w)$$

(c) From the PDF $f_W(w)$, calculating the moments is straightforward.

$$E[W] = \int_{-\infty}^{\infty} w f_W(w) \, dw = (1/2) \int_{-\infty}^{\infty} w f_X(w) \, dw = E[X]/2$$

The second moment is

$$E[W^2] = \int_{-\infty}^{\infty} w^2 f_W(w) \, dw = (1/2) \int_{-\infty}^{\infty} w^2 f_X(w) \, dw = E[X^2]/2$$

The variance of $W$ is

$$\text{Var}[W] = E[W^2] - (E[W])^2 = E[X^2]/2 - (E[X]/2)^2$$

$$= (1/2) \text{Var}[X] + (E[X])^2/4$$