Final Exam

1. (20 points)
   
   (a) State the criterion for determining whether an additive plant perturbation affects the stability of the closed-loop system.

   \[
   \text{WHERE } G_a(s) = G(s) + A(s)
   \]

   \[
   |A(j\omega)| < |1 + G(j\omega)| \quad \forall \omega
   \]

   (b) State the criterion for determining whether an multiplicative plant perturbation affects the stability of the closed-loop system.

   \[
   \text{WHERE } G_m(s) = G(s)(1 + M(s))
   \]

   \[
   |M(j\omega)| < |1 + \frac{1}{G(j\omega)}| \quad \forall \omega
   \]
(c) Use the appropriate criterion to determine whether the unmodelled pole $2/(s + 2)$ of the plant affects the stability of the nominal plant:

$$G(s) = \frac{30}{(s + 1)(s + 5)}.$$ 

We apply the multiplicative method. First we find $M(s)$:

$$M(s) + 1 = \frac{2}{s+2},$$

so

$$M(s) = \frac{2}{s+2} - 1 = \frac{-s - 2 + 2}{s+2} = -\frac{s}{s+2}.$$

$$1 + \frac{1}{G(s)} = 1 + \frac{(s+1)(s+5)}{30}$$

$$= \frac{30 + (s+1)(s+5)}{30} = \frac{s^2 + 6s + 35}{30}$$

$$= \frac{s^2 + 6s + 35}{30} = \frac{s^2 + 6s + 35}{30},$$

for $s^2 + 6s + 35 = s^2 + 25\omega_n^2s + \omega_n^2$.

$$\omega_n = \sqrt{35} \approx 6.$$

Our analysis suggests it is OK to ignore the pole*.

* When it's this close we'd definitely want to use MATLAB.

$$20 \log_{10} \left| \frac{1}{G(s)} \right| \approx -10$$

$$20 \log_{10} \frac{35}{30} \approx 2$$

\[ \text{(dB)} \]
2. (15 points) The Bode plot for the open loop transfer function

\[ G(s) = \frac{20}{s^2} \]

is shown below. The values of \( \alpha, \omega_m, \) and \( \tau \) are required for a lead compensator

\[ G_c(s) = \frac{(1 + \alpha \tau s)}{\alpha (1 + \tau s)} \]

to increase the phase margin.

continued on next page...
(a) Suppose $\zeta = 0.2$ is required. Determine the appropriate value for $\alpha$.

$$\phi_{pm} = \frac{5}{0.01} = \frac{0.2}{0.01} = 20 \text{ deg}$$

$$\sin \phi_{pm} = \frac{\alpha - 1}{\alpha + 1} \quad \Rightarrow \quad \frac{\alpha - 1}{\alpha + 1} = 2/3 \alpha$$

$$\frac{1}{3} + 1 = \frac{2}{3} \alpha \quad \Rightarrow \quad \alpha = \frac{4/3}{2/3} = 2$$

(b) Suppose $\alpha = 20$. From the Bode plot, determine the value of $\omega_m$. (Justify your answer.)

$\omega_m$ is the frequency at which the magnitude is $-10 \log_{10} \alpha$

$$-10 \log_{10} 20 = -10 \left( \log_{10} 10 + \log_{10} 2 \right) \approx -15 \text{ rad/s}$$

From graph, $\omega_m \approx 10 \text{ rad/s}$

(c) For this value of $\omega_m$, what is the value of $\tau$?

$$\tau = \frac{1}{\omega_m \sqrt{\alpha}} \approx \frac{1}{10 \sqrt{20}} = \frac{1}{40 \sqrt{5}} \text{ using } \alpha \text{ from part (b)}$$

$$\approx \frac{1}{\omega_m \sqrt{\alpha}} \approx \frac{1}{10 \sqrt{2}} \text{ using } \alpha \text{ from part (a)}$$
3. (15 points) Consider the system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]

with output

\[
y = [1 \\ 0] x.
\]

Calculate the controllability matrix \( P_c \) and the observability matrix \( P_o \). State whether the system is controllable or not and whether it is observable or not.

\[
P_c = \begin{bmatrix} B & AB \\ C & CA \end{bmatrix} \quad (n = 2)
\]

\[
= \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}. \quad \text{det} \ P_c = 0 \quad \text{so} \quad P_c \ \text{does NOT HAVE FULL RANK so the sys. IS NOT CONTROLLABLE}
\]

\[
P_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 0 \end{bmatrix}. \quad \text{det} \ P_o = 0 \quad \text{so} \quad P_o \ \text{does NOT HAVE FULL RANK so the system is NOT OBS.}
4. (15 points) Consider the system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u
\]

with output

\[ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x + u. \]

Determine the poles of the observer if the observer gain is \( L = [0 \ 1]^T \).

\[
A - LC = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}
\]

\[
= \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}
\]

\[
| sI - (A - LC) | = \begin{vmatrix} s - 3 & 0 \\ 0 & s - 2 \end{vmatrix} = (s - 2)(s - 3). \]

Poles are \( 2, 3 \).
5. (15 points) Consider the system

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix} 3 & 0 \\
1 & 2
\end{bmatrix} \begin{bmatrix} x_1 \\
x_2
\end{bmatrix} + \begin{bmatrix} 0 \\
1
\end{bmatrix} u
\]

with output

\[ y = [1 \ 1] x + u. \]

Find the transfer function for the system.

\[
G(s) = C(sI - A)^{-1}B + D
\]

\[
sI - A = \begin{bmatrix} s - 3 & 0 \\
1 & s - 2
\end{bmatrix}
\]

\[
|sI - A| = (s - 2)(s - 3)
\]

so \((sI - A)^{-1} = \begin{bmatrix} \frac{1}{s - 3} & 0 \\
\frac{1}{(s - 2)(s - 3)} & \frac{1}{s - 2}
\end{bmatrix}\)

and

\[
G(s) = \begin{bmatrix} 1 & 1 \\
\frac{1}{s - 3} & 0
\end{bmatrix} \begin{bmatrix} 1 \\
\frac{1}{(s - 2)(s - 3)} & \frac{1}{s - 2}
\end{bmatrix} + \begin{bmatrix} 0 \\
1
\end{bmatrix}
\]

\[
= \begin{bmatrix} 1 & 1 \\
\frac{1}{s - 2}
\end{bmatrix} + \frac{1}{s - 2}
\]

\[
= \frac{1}{s - 2} + \frac{s - 2}{s - 2} = \frac{s - 1}{s - 2}
\]
6. (15 points) Consider the negative unity feedback system with open-loop transfer function

\[ G(s) = \frac{Ks + 1}{s(s + p_1)(s + p_2)} \]

If \(1 < p_1 < 5\), \(-1 < p_2 < 2\), and \(K = 3\), identify the four worst-case polynomials for the closed-loop transfer function and determine whether the system is stable for all variations of the parameters within the indicated ranges.

\[ G_{ac}(s) = \frac{Ks + 1}{s(s + p_1)(s + p_2) + Ks + 1} \]

\[ = \frac{Ks + 1}{s^3 + (p_1 + p_2)s^2 + (p_1p_2 + K)s + 1} \]

Given \(p_1 \in [1, 5]\), \(p_2 \in [-1, 2]\), \(K = 3\), we have

\[ 0 \leq p_1 + p_2 \leq 7 \quad \text{for} \quad a_2 \]

\[ -2 \leq p_1p_2 + K \leq 13 \quad \text{for} \quad a_1 \]

\[ 1 \leq 1 \leq 1 \quad \text{for} \quad a_0 \]

\[ \Rightarrow \quad \alpha_2 = 0, \quad \beta_2 = 7 \]

\[ \alpha_1 = -2, \quad \beta_1 = 13 \]

\[ \alpha_0 = 1, \quad \beta_0 = 1 \]

\[ q_1(s) = s^3 + 0s^2 + 13s + 1 \]

\[ q_2(s) = s^3 + 7s^2 - 2s + 1 \]

\[ q_3(s) = s^3 + 7s^2 + 13s + 1 \]

\[ q_4(s) = s^3 + 0s^2 - 2s + 1 \]

Because of the negative coefficient in \(q_2\) and \(q_4\), we predict instability.
We can express the Routh array as

\[
\begin{array}{ccc}
S^3 & 1 & a_1 \\
S^2 & a_2 & 1 \\
S & -\frac{1}{a_2} (1-a_1 a_2) \\
S^0 & 1 \\
\end{array}
\]

which indicates that we need \( a_2 > 0 \). This is not the case for \( q_1 \) and \( q_4 \) so stability is not guaranteed.

From row 3 we see that we would need \( a_1 a_2 > 1 \) so we would need \( a_1 > \frac{1}{a_2} \). This is not the case for \( q_2 \) and \( q_4 \), so again we see that stability is not guaranteed.