Solution to HW4

AP4.3 Instead of a constant or step reference input, we are given, in this problem, a more complicated reference path,

\[ r(t) = (2 - t - t^2/2)u(t). \]  

(a) To determine the steady state error with zero disturbance, we determine the transfer function from \( R(s) \) to \( Y(s) \). Using block diagram transformations we obtain

\[ T(s) = \frac{Y(s)}{R(s)} = \frac{5(s + 1)}{s^3 + 3s^2 + 5s + 5}. \]  

Then taking the Laplace transform of \( r(t) \) to obtain

\[ R(s) = \frac{2}{s} - \frac{1}{s^2} + \frac{2}{2s^3} = \frac{2s^2 - s + 1}{s^3} \]  

and noting that

\[ E(s) = R(s) - Y(s) = R(s)(1 - T(s)) \]

we have

\[ E(s) = \left( \frac{2s^2 - s + 1}{s^3} \right) \left( \frac{s^3 + 3s^2}{s^3 + 3s^2 + 5s + 5} \right) = \frac{(2s^2 - s + 1)(s + 3)}{s(s^3 + 3s^2 + 5s + 5)} \]

so we obtain

\[ e_{ss} = \lim_{s \to 0} sE(s) = \frac{3}{5}. \]  

(b) Next we are asked to plot the error for this path over the time interval 0 to 10 seconds. To obtain the required plot, we note that we can factor \( 1/s \) from the expression for \( E(s) \) so that we can use the MATLAB command \texttt{step} to generate the response using the following MATLAB code:

\begin{verbatim}
>> sys = tf(conv([2 -1 1],[1 3]),conv([1 0],[1 3 5 5]))
Transfer function:
  2 s^3 + 5 s^2 - 2 s + 3
-------------------------
 s^4 + 3 s^3 + 5 s^2 + 5 s

>> sys1 = tf([2 5 -2 3],[1 3 5 5])
Transfer function:
  2 s^3 + 5 s^2 - 2 s + 3
-------------------------
 s^3 + 3 s^2 + 5 s + 5

>> t = [0:.01:10]'; u = 0*t;
>> y = step(sys1,t);
\end{verbatim}
The resulting plot, shown below, indicates a steady state error of approximately 0.6, agreeing with our answer in part (a).

(c) Now for zero input \( r(t) = 0 \), we are asked to find the steady state error due to a step disturbance \( D(s) = 1/s \). We first redraw the block diagram as
from which we can use block diagram transformations to find the transfer function from $-D(s)$ to $Y(s)$, which is

$$T(s) = \frac{s}{s^3 + 3s^2 + 5s + 5}$$

so

$$e_{ss} = \lim_{s \to 0} sT(s)(-D(s)) = -\lim_{s \to 0} T(s) = 0.$$  \hspace{1cm} (8)

(d) Finally we use MATLAB to plot the error due to the disturbance as we did in part (b), using the commands below to generate the plot shown.

```matlab
>> sys2 = tf([1 0],[1 3 5 5])
Transfer function:
    s
-------------
s^3 + 3s^2 + 5s + 5

>> t = [0:.01:10]'; u = 0*t;
>> y = -step(sys2,t);
>> plot(t,y)
>> grid
>> xlabel('Time(s)'); ylabel('e(t)');
>> title('Error due to disturbance in AP4.3d');
>> print -deps ap4_3d.eps
```
In this problem we choose a gain $K$ to ensure that the steady state error due to a ramp input signal $v(t)$ does not exceed 0.1, and then plot the error as a function of time for a ramp disturbance input applied for the time period 0 to 5 seconds.

(a) Finding an expression for the steady state error due to a ramp input is straightforward. The Laplace transform of the input is $V(s) = 1/s^2$. Standard block diagram transformations allow us to determine the transfer function from $V(s)$ to $\omega(s)$, which is

$$T(s) = \frac{KK_m}{R_aJs^2 + K_mK bs + KK_mK_t}$$

and as in problem AP4.3a, we find the Laplace transform of the error to be

$$E(s) = V(s) - K_t\omega(s) = V(s)(1 - K_tT(s))$$

so

$$E(s) = V(s)\left[\frac{R_aJs + K_mK_b}{R_aJs^2 + K_mK bs + KK_mK_t}\right]$$

and

$$e_{ss} = \lim_{s \to 0} sE(s) = s \lim_{s \to 0} \left(\frac{1}{s^2}\right)\left[\frac{R_aJs + K_mK_b}{R_aJs^2 + K_mK bs + KK_mK_t}\right] = \frac{K_b}{K}.$$ 

Since $K_b = 0.1$, to keep the steady state error less than 0.1, we need $K > 1$. 

\[ \text{Error due to disturbance in AP4.3d} \]
(b) Proceeding as in AP4.3c, we find the transfer function from $D(s)$ to $\omega(s)$ to be

$$T(s) = \frac{K_m s}{R_a J s^2 + K_m K_b s + K K_m K_t}.$$  \hfill (13)

We are given the disturbance

$$d(t) = \begin{cases} -t & 0 \leq t \leq 5 \\ 0 & 5 < t \leq 10 \end{cases} \hfill (14)$$

where the minus sign comes from the fact that $D(s)$ is input to the summer in the block diagram with a negative sign. This input has the Laplace transform

$$D(s) = \begin{cases} -1/s^2 & 0 \leq t \leq 5 \\ 0 & 5 < t \leq 10 \end{cases}. \hfill (15)$$

The steady state error due to a ramp input of infinite duration may or may not be (approximately) attained when the disturbance is applied for only a finite time. Whether it is attained will depend on the value we choose for $K$, which affects the speed of the system response. However, using the steady state error corresponding to a disturbance of infinite duration will give us an upper bound on the error due to a temporary disturbance. Using the given parameter values and our chosen gain, $K = 2$, we obtain

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} sT(s)D(s) = -\frac{1}{2}. \hfill (16)$$

We’ll plot the response to the temporary disturbance input using the command \texttt{lsim} as shown below. Note that the state space representation of the system has a state vector of length 2 so the initial condition in the \texttt{lsim} command is also a vector of length 2. We assume zero initial conditions.

```matlab
>> t = [0:.01:10]’;
>> u(1:501) = -t(1:501);
>> u(502:1001) = zeros(500,1);
>> x0 = [0 0]’;
>> sys3 = tf([10 0],[1 10 20])
```

Transfer function:

$$\frac{10 s}{s^2 + 10 s + 20}$$

```matlab
>> ss_sys3 = ss(sys3)
a =
   x1     x2
x1  -10    -5
x2     4     0
```
\begin{verbatim}
b =
   u1
   x1 4
   x2 0

c =
   x1  x2
   y1  2.5  0

d =
   u1
   y1  0

Continuous-time model.
>> [y,t,x] = lsim(ss_sys3,u,t,x0);
>> plot(t,y)
>> xlabel('Time(s)'); ylabel('e(t)');
>> title('Error due to disturbance in AP4.4b');
>> grid
>> print -deps ap4_4b.eps

We obtain the following plot. Notice that before the disturbance goes to zero at 5 seconds, the error has leveled off to the steady state value of -1/2 that we found above.
\end{verbatim}
AP4.6 In this problem we analyze the lead network circuit shown in Figure AP4.6, page 217.

(a) To determine the transfer function from \( V(s) \) to \( V_0(s) \) we note that we have a simple voltage divider with \( v_0 \) taken across the impedance \( Z_2 = R \) and the total impedance being the series connection of \( Z_1 \) and \( Z_2 \) where \( Z_1 \) is the impedance resulting from the parallel connection of the capacitor and resistor. Accordingly,

\[
Z_1 = \frac{R \left( \frac{1}{Cs} \right)}{R + \left( \frac{1}{Cs} \right)} = \frac{R}{RCs + 1}. \tag{17}
\]

and the transfer function is then

\[
G(s) = \frac{V_0(s)}{V(s)} = \frac{Z_2}{Z_1 + Z_2} = \frac{RCs + 1}{RCs + 2}. \tag{18}
\]

(b) To obtain the sensitivity of \( G(s) \) with respect to the capacitance \( C \), we apply equation (4.16) of the text which tells us that

\[
S^G_C = S^N_C - S^D_C \tag{19}
\]

where \( N(s) \) and \( D(s) \) are the numerator and denominator, respectively, of \( G(s) \). We then have

\[
S^N_C = \frac{\partial N}{\partial C} \frac{C}{N} = Rs \frac{C}{RCs + 1}. \tag{20}
\]
and
\[ S_C^D = \frac{\partial D}{\partial C} D = Rs \frac{C}{RCs + 2} \] (21)
so
\[ S_C^G = \frac{RCs}{(RCs + 1)(RCs + 2)}. \] (22)

(c) The transient response with \( V(s) = 1/s \) can be determined analytically using a partial fraction expansion of \( V_0(s) = V(s)G(s) \) and inverse transforming. Since
\[ V_0(s) = sRCs + 1 \] (23)
we solve
\[ \frac{A}{s} + \frac{B}{RCs + 2} = \frac{RCs + 1}{s(RCs + 2)} \] (24)
for \( A \) and \( B \) to obtain \( A = 1/2 \) and \( B = RC/2 \), so
\[ v_0(t) = \mathcal{L}^{-1} \{ V_0(s) \} = \frac{1}{2} \left( 1 + e^{-\frac{2}{RC}t} \right) u(t). \] (25)

We can plot this using MATLAB by plotting \( v_0(\tilde{t}) \) vs. \( \tilde{t} = \frac{t}{RC} \). We find and plot \( v_0(t) \) using the commands
\[
\text{>> for index = 1:1001, v0(index) = (1+exp(-2*t(index)))/2;end}
\text{>> plot(t,v0)}
\text{>> xlabel('t/(RC)'); ylabel('v_0');}
\text{>> title(' Step response for problem AP4.6c');}
\text{>> print -deps ap4_6c.eps}
\]
and obtain the plot below.
DP4.2 This problem asks us to choose controller gain $K_1 K$ in order to reject step disturbances while maintaining acceptable transient response. We are told that keeping $K_1 K < 35$ will achieve acceptable transient response, so we need only concern ourselves with the disturbance rejection question. The block diagram relating $T_d(s)$ to $\theta(s)$ has $G(s)$ in the forward path and $K_1 K/s$ in the backward path. The transfer function is thus

\[
\frac{\theta(s)}{T_d(s)} = \frac{s}{s^3 + 4s^2 + 9s + K_1 K}.
\]  

(26)

We can now use MATLAB to plot the response of the system to a step input using the following commands.

\[
\text{>> KK1} = [1 6 12 18 24]
\]

\[
\text{KK1} =
\begin{bmatrix}
1 & 6 & 12 & 18 & 24
\end{bmatrix}
\]

\[
\text{>> for index = 1:5, sys(index,:) = tf([1 0],[1 4 9 KK1(index)])}
\]

end
Transfer function from input to output...

\[
\begin{align*}
\text{#1: } & \quad \frac{s}{s^3 + 4s^2 + 9s + 1} \\
\text{#2: } & \quad \frac{s}{s^3 + 4s^2 + 9s + 6} \\
\text{#3: } & \quad \frac{s}{s^3 + 4s^2 + 9s + 12} \\
\text{#4: } & \quad \frac{s}{s^3 + 4s^2 + 9s + 18} \\
\text{#5: } & \quad \frac{s}{s^3 + 4s^2 + 9s + 24}
\end{align*}
\]

\[>> \text{for index = 1:5, } y(:,\text{index})=\text{step(sys(index),t); end;}\]
\[>> \text{plot(t,y)}\]
\[>> \text{legend('K_1K = 1','K_1K = 6','K_1K = 12','K_1K = 18','K_1K = 24')}\]
\[>> \text{grid}\]
\[>> \text{xlabel('Time (s)'); ylabel('\theta');}\]
\[>> \text{title('Step Response for DP4.2')}\]
\[>> \text{print -depsc dp4_2.eps}\]

We see from the plot below that choosing $K_1K$ in the range of 6 to 12 damps the disturbance reasonably quickly without introducing too much oscillation.
### DP4.3

For the control system shown in Figure DP4.3, p. 219 of the text, we are asked to find a range of values of the gain $K_1$ for which the steady state error will be less than 1%. We are then asked to determine values of $K_1$ and $K$ so that the steady state error due to a wind disturbance $d(t) = 2t \text{ mrad/s}$ for $0 \leq t < 5$ seconds is less than 0.1 mrad.

**Solution:** Following the same procedure as in earlier problems we find that the transfer function from $\omega_d(s)$ to $\omega(s)$ is

$$T(s) = \frac{\omega(s)}{\omega_d(s)} = \frac{K}{s^2 + 5s + KK_1}.$$  \hfill (27)

The problem is not well posed in that it does not say within 1% of what the steady state error should be. Let’s assume that they mean that the steady state error should be less than 1% of absolute value of the magnitude $M$ of a step input. Using $E(s) = \omega_d(s) - \omega(s) = \omega_d(s)(1 - T(s))$ we find

$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} s \frac{M}{s} \left( \frac{s^2 + 5s + K(K_1 - 1)}{s^2 + 5s + KK_1} \right) = \frac{MK(K_1 - 1)}{K_1 K}.$$ \hfill (28)

Then $e_{ss} < 0.01|M|$ requires

$$K(K_1 - 1) < 0.01K_1 K.$$ \hfill (29)
or

$$0.99KK_1 < K \quad (30)$$

so we need $K_1 < 1.01$.

Next we consider the disturbance input. We notice that the wind disturbance occurs only for a period of 5 seconds and then stops. We'll find the steady state error ignoring the transient nature of the disturbance and then make the (not unreasonable in this case) assumption that the steady state error due to the temporary disturbance will be upper bounded by that of a permanent disturbance.

With $D(s) = 2 \times 10^{-3}/s^2$ and (proceeding as in earlier problems)

$$E(s) = \frac{s}{s^2 + 5s + K_1K}(-D(s)) \quad (31)$$

we obtain

$$e_{ss} = \lim_{s \to 0} sE(s) = \frac{-2 \times 10^{-2}}{K_1K} \quad (32)$$

so to make this less in magnitude than $0.1 \times 10^{-3}$ requires $K_1K > 20$. To be sure that our assumption is reasonable, we can use MATLAB to plot the response to the disturbance, just as we did in problem AP4.4. With $K_1K = 30$, we use the following MATLAB commands to generate a plot of the disturbance response. Notice that only the steady state value and not the transient value of the disturbance response is within the required range. If we wished to maintain $\omega < 0.1$ mrad for all time we would need a much more conservative design.

```
>> t = [0:.01:10]';
>> u(1:501) = 2*t(1:501);
>> u(502:1001) = zeros(500,1);
>> x0 = [0 0]';

>> sys = tf([1 0],[1 5 30])

Transfer function:
    s
----------
   s^2 + 5 s + 30

>> ss_sys=ss(sys)

a =
   x1   x2
   x1  -5  -7.5
   x2   4   0

b =
   u1
```
\[ x_1 \quad 1 \\
\quad x_2 \quad 0 \\
\]

c = 
\[
\begin{array}{c}
x_1 \\
y_1 
\end{array} \quad 
\begin{array}{c}
x_2 \\
1 \\
0 
\end{array}
\]

d = 
\[
\begin{array}{c}
u_1 \\
y_1 
\end{array} 
\begin{array}{c}
1 \\
0 
\end{array}
\]

Continuous-time model.
\[
[y,t,x] = \text{linsim}(ss\_sys,u*1e-3,t,x0); \\
\text{plot}(t,y*1000) \\
\text{figure}(1) \\
\text{grid} \\
\text{xlabel('Time (s)'); ylabel('\omega (mrad/s)');} \\
\text{title('Plot of disturbance response for DP4.3')} \\
\text{figure(1)} \\
\text{print -deps dp4_3b.eps}
\]
MP4.6 We are asked to compute and compare the open- and closed-loop responses to a step disturbance for the given system. With the given parameter values we use the following MATLAB code to produce the plot below. We see that closing the loop reduces the effect of the disturbance to approximately 1/10 of its open-loop value.

```matlab
>> t = [0:.01:10];
>> olsys = tf([1],[1 0.9 5]);
>> oly = step(olsys,t);
>> clsys = tf([1],[1 0.9 55]);
>> cly = step(clsys,t);
>> plot(t,oly,'-.',t,cly,'-')
>> legend('Open Loop','Closed Loop');
>> xlabel('Time (s)'); ylabel('	heta (rad)');
>> grid
>> title('Response to Step Disturbance in MP4.6')
>> print -deps mp4_6.eps
```
Response to Step Disturbance in MP4.6

- Open Loop
- Closed Loop