Solution to HW8

AP10.1 We are given a unity negative feedback system with a compensator having transfer function $G_c(s)$ preceding the plant in the forward path. The plant transfer function is

$$G(s) = \frac{1}{s(s + 1)(s + 4)}.$$  

In part (a) we are asked to determine a gain $K$ such that $G_c(s) = K$ satisfies the requirement that the overshoot in response to a step input should be less than 13%. We are then asked to determine the resulting 2% settling time for the closed–loop system. In part (b) we are asked to design a lead network to reduce the settling time to 3 seconds.

Solution:

(a) The percent overshoot is given by

$$P.O. = 100e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}.$$  

Thus we solve for $\zeta$ as follows:

$$\frac{13}{100} = e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

so

$$\ln 13 - \ln 100 = -\frac{\pi\zeta}{\sqrt{1-\zeta^2}}$$

and thus

$$(\ln 13 - \ln 100)^2(1 - \zeta^2) = \pi^2\zeta^2$$

so finally

$$\zeta = -\frac{\ln 13 - \ln 100}{\pi^2 + (\ln 13 - \ln 100)^2} = 0.5118$$

where the positive square root was chosen since $\zeta$ must be nonnegative. Increasing $\zeta$ decreases the percent overshoot so this is a lower bound on acceptable values of $\zeta$. Let’s take $\zeta = 0.55$ to give ourselves a margin of error. This corresponds to a percent overshoot of 12.6.

The closed–loop transfer function is

$$T(s) = \frac{K}{s^3 + 5s^2 + 4s + K}$$

so we can use the root locus method to determine the value of $K$ corresponding to $\zeta = 0.55$. Using the MATLAB command `rlocus(i,[1 5 4 0])` corresponding to the root locus for

$$1 + \frac{K}{s^3 + 5s^2 + 4s} = 0$$

obtained by dividing the denominator of $T(s)$ by the terms not containing the gain $K$ we find that $K = 2.26$ corresponds to $\zeta = .559$ so we choose $K = 2.25$. We then
determine the step response of the corresponding closed-loop system and verify that the percentage overshoot is an acceptable 11.44%.

The MATLAB commands to find $\zeta$ and $K$ are shown in the transcript below. To determine the settling time I generated a step response for a time vector (0 to 20 seconds) that was large enough to produce the initial overshoot followed by some additional oscillation. I then used the `find` command to identify the indices of the values of the output $y$ for which the response was not within 2% of the steady-state value of 1. Adding one to the largest such index gives the index of the settling time. The settling time I found was 8.25 seconds.

Here’s the MATLAB script, followed by the plots generated.

```matlab
% Solve AP10.1(a)

zeta = -(log(13)-log(100))/sqrt(pi^2+(log(13)-log(100))^2)
sys = tf(1,[1 5 4 0])

rlocus(sys)
title('RootLocus for AP10.1a')
print -deps ap10_1a_rlocus.eps

K = input('Enter K chosen from root locus for zeta calculated above: ')
clsys=feedback(K*sys,1)
t=[0:.01:20];
y=step(clsys,t);figure(2);plot(t,y);grid
title(['Step response for closed loop system with gain K = ',num2str(K)])
xlabel('Time (s)')
print -deps ap10_1a_step.eps

disp('Maximum overshoot:')
max(y)
indices = find(abs(y-ones(size(y)))>0.02);
disp('Settling time:')
Ts = t(max(indices)+1)
```

(b) Next, we are asked to use a lead compensator with transfer function

$$G_c(s) = \frac{1 + \alpha \tau s}{\alpha (1 + \tau s)} = \frac{s + z}{s + p}$$

to reduce the settling time to under 3 seconds while maintaining the percent overshoot less than 13%. We do this in the following steps.

1. We determine the required $\omega_n$ from the expression for $T_s$ ($T_s = 4/(\zeta \omega_n)$). Using again $\zeta = 0.55$ we find that for $T_s = 3$ we obtain $\omega_n = 2.4$ rad/s.
2. Using MATLAB to generate a Bode plot for the closed-loop system for

$$G(s) = \frac{2.25}{s^3 + 5s^2 + 4s}.$$  

we see that the 3dB-bandwidth attained is only 0.85 rad/s. We use the commands
Figure 1: Root locus plot for determining $K$ corresponding to $\zeta = 0.55$

$$K = [2.25]$$

$$s1 = \text{feedback}(\text{tf}([K],[1 5 4 0]),1)$$

$$s3 = \text{feedback}(\text{tf}([3*K],[1 5 4 0]),1)$$

$$s5 = \text{feedback}(\text{tf}([5*K],[1 5 4 0]),1)$$

$$s7 = \text{feedback}(\text{tf}([7*K],[1 5 4 0]),1)$$

To test higher gains until we find one that achieves the required bandwidth. The Bode plot is shown in Figure 3. The system $s7$ whose Bode plot has acceptable bandwidth has gain $7 \times 2.25 = 15.75$ so we choose $K = 16$.

3. We use the `margin` command to find the phase margin for the open-loop system with gain $K = 16$ as shown below:

```matlab
>> [Gm,Pm,wcg,wcp]=margin([16],[1 5 4 0])
```

$$Gm =$$

1.2500
Figure 2: Step response for controlled system with $G_c(s) = K = 2.25$

Step response for closed loop system with gain $K = 2.25$

$P_m = 5.2057$

$w_{cg} = 2.0000$

$w_{cp} = 1.7852$
4. We calculate the required phase to be added using the lead compensator. For $\zeta = .55$, the rule of thumb presented in the text in (9.58), page 492, is that $\phi_{pm}$ should be $100\zeta$ degrees so we need $55 - (-5) = 50$ degrees.

5. Calculate $\alpha$ from (10.11), p. 559. We obtain $\alpha = 7.5$.

6. Find the new 0 dB crossover frequency $\omega_m$ by identifying the frequency on the Bode plot where the gain is $-10\log \alpha$ dB. This gain is $-10\log 7.5 = -8.75$. The corresponding frequency is approximately 2.9 rad/s.

7. Determine the pole and zero of the compensator.

$$p = \omega_m \sqrt{\alpha} = 7.94$$

and

$$z = p/\alpha = 1.06.$$ 

8. Verify that the required phase margin of 55 degrees has been achieved. The transcript below shows that we achieved a phase margin of 77.5 degrees so our phase margin is acceptable.

$$\gg \ Gc = \text{tf}([1 \ 1.06],[1 \ 7.94])$$
Transfer function:
\[
s + 1.06 \\
--------- \\
s + 7.94
\]

>> G=tf(16,[1 5 4 0])

Transfer function:
\[
16 \\
---------- \\
s^3 + 5s^2 + 4s
\]

>> [Gm,Pm,wcg,wcp]=margin(series(Gc,G));
   >> Pm

Pm =

77.4550

9. Plot the step response of the resulting closed-loop system to determine whether it meets the design requirements, then increase the compensator gain until it does. Increasing the gain by a factor of 3 to yielded a closed-loop system with 9.3% overshoot and \( T_s = 2.85 \) seconds, thereby meeting the design requirements. The step responses obtained in this step are shown in Figure 4.

The final compensator design is

\[
G_c = \frac{48(s + 1.06)}{(s + 7.94)}
\]

for the plant

\[
G(s) = \frac{1}{s(s + 1)(s + 4)}
\]

It should be noted that this design method does not produce an optimal compensator design. It simply produces a design that meets the design specifications.

**AP10.2** For the same system, we are asked to design a lag network to achieve percent overshoot less than 13% and steady-state error for a unit ramp input less than 0.125. We are then asked to find the percent overshoot and 2% settling time for the resulting closed-loop system.

**Solution:** Note that this is solution is a way to solve the problem. It is not the only way to approach the problem.

The lag network will have transfer function

\[
G_c(s) = \frac{K(s + z)}{(s + p)}
\]
so the closed-loop transfer function will be

\[ T(s) = \frac{K(s + z)}{s^4 + (5 + p)s^3 + (4 + 5p)s^2 + (4p + K)s + Kz} \]

so the error will be

\[ R(s) - Y(s) = R(s)[1 - T(s)] = R(s) \frac{s^4 + (5 + p)s^3 + (4 + 5p)s^2 + 4ps}{s^4 + (5 + p)s^3 + (4 + 5p)s^2 + (4p + K)s + Kz} \]

so the steady state error for a unit ramp \( R(s) = 1/s^2 \) is

\[ e_{ss} = \lim s \left( \frac{1}{s^2} \right) \frac{s^4 + (5 + p)s^3 + (4 + 5p)s^2 + 4ps}{s^4 + (5 + p)s^3 + (4 + 5p)s^2 + (4p + K)s + Kz} = \frac{4p}{Kz} \]

If we choose \( K = 2.25 \) as in AP10.1, we then have the condition

\[ \frac{4p}{2.25z} = \frac{1.78p}{z} < 0.125 \]
so we now have the constraint \( p < 0.07z \). We determine the value of \( z \) and \( p \) in the following steps.

1. We locate the frequency on the Bode plot where the phase margin is \( 60 = 55 + 5 \) degrees as specified in the instructions on page 581 of the text. This frequency is \( 0.438 \) rad/s.

2. We place the \( z \) at \( 0.438/10 = 0.0438 \) rad/s (one decade below the frequency found in the previous step), again as instructed on page 581.

3. We set \( p = 0.06(0.438) = 0.026 < 0.07(0.438) \).

4. Using MATLAB and finding the step response of the closed-loop system with this compensator to be inadequate (too slow), we reduce \( z \) until the behavior is satisfactory. For \( z = 0.01 \) we obtain 12.26\% overshoot and error for a unit ramp input

\[ e_{ss} = 4p/Kz = 1.0667 < 0.125. \]

**AP10.3** For the same system, we are asked to design a PI controller so that the closed-loop system has percent overshoot less than 13\% and steady state error to a unit ramp of less than 0.125.

**Solution:** The transfer function for the PI controller is

\[ G_c(s) = \frac{KPs + K_I}{s}. \]

With this controller, the closed loop transfer function will be

\[ T(s) = \frac{KPs + K_I}{s^4 + 5s^3 + 4s^2 + KPs + K_I} \]

and the error will be

\[ R(s) - Y(s) = R(s)[1 - T(s)] = R(s)\frac{s^4 + 5s^3 + 4s^2}{s^4 + 5s^3 + 4s^2 + KPs + K_I} \]

so the steady state error for a unit ramp \( R(s) = 1/s^2 \) is

\[ e_{ss} = \lim s \left( \frac{1}{s^2} \right) \frac{s^4 + 5s^3 + 4s^2}{s^4 + 5s^3 + 4s^2 + KPs + K_I} = 0 \]

so any \( K_P \) and \( K_I \) that result in a stable system will satisfy the steady state error criterion.

From AP10.1 we know that a gain of 2.25 will result in a closed-loop system with less than 13\% overshoot so we start with \( K_P = 2.25 \) and adjust \( K_I \) to obtain satisfactory behavior. By generating the step response in MATLAB for the closed-loop system with different values of \( K_I \) I found that \( K_I = 1 \) lead to oscillations of increasing amplitude in the step response showing that this was too high a gain. I then tried \( K_I = 0.1 \) and found that this still did not achieve acceptable system behavior. Finally I tried \( K_I = 0.01 \), which achieved percent overshoot of 12.3\%. Here’s the MATLAB transcript.
>> sys = tf(1,[1 5 4 0])

Transfer function:
    1
--------------
  s^3 + 5 s^2 + 4 s

>> csys = tf([2.25 1],[1 0])

Transfer function:
  2.25 s + 1
---------
    s

>> clsys = feedback(series(csys,sys),1)

Transfer function:
   2.25 s + 1
--------------
 s^4 + 5 s^3 + 4 s^2 + 2.25 s + 1

>> y=step(clsys,t);plot(t,y);grid
>> csys = tf([2.25 .1],[1 0])

Transfer function:
  2.25 s + 0.1
---------
    s

>> clsys = feedback(series(csys,sys),1)

Transfer function:
  2.25 s + 0.1
---------
 s^4 + 5 s^3 + 4 s^2 + 2.25 s + 0.1

>> y=step(clsys,t);plot(t,y);grid
>> csys = tf([2.25 .01],[1 0])

Transfer function:
  2.25 s + 0.01
---------
    s

>> clsys = feedback(series(csys,sys),1)

Transfer function:
   2.25 s + 0.01
---------
 s^4 + 5 s^3 + 4 s^2 + 2.25 s + 0.1

>> y=step(clsys,t);plot(t,y);grid
Transfer function:
\[
\frac{2.25 s + 0.01}{s^4 + 5 s^3 + 4 s^2 + 2.25 s + 0.01}
\]

```matlab
>> y=step(clsys,t);plot(t,y);grid
>> max(y)
```
```
ans =
    1.1236
```